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Central charge and quasihole scaling dimensions from model wavefunctions: toward relating Jack wavefunctions to $\mathcal{W}$-algebras

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Abstract

We present a general method for obtaining the central charge and the quasihole scaling dimension directly from ground-state and quasihole wavefunctions. Our method applies to wavefunctions satisfying specific clustering properties. We then use our method to examine the relation between Jack symmetric functions and certain $\mathcal{W}$-algebras. We add substantially to the evidence that the (k, r) admissible Jack functions correspond to correlators of the conformal field theory $\mathcal{W}_k(k+1, k+r)$ by calculating the central charge and scaling dimensions of some of the fields in both cases and showing that they match. For the Jacks described by unitary $\mathcal{W}$-models, the central charge and quasihole exponents match those previously obtained from analyzing the physics of the edge excitations. For the Jacks described by non-unitary $\mathcal{W}$-models the central charge and quasihole scaling dimensions obtained from the wavefunctions differ from those obtained from the edge physics, which instead agree with the 'effective' central charge of the corresponding $\mathcal{W}$-model.

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1. Introduction

In the lowest Landau level in symmetric gauge [1, 2], wavefunctions can be thought of as single-valued analytic functions of complex variables. As a result, many powerful mathematical tools can be brought to bear on the study of lowest Landau level physics. In particular, the power of conformal field theory [3] has been useful for understanding fractional quantum Hall wavefunctions. Starting with the work of Moore and Read [4], it was realized that correlators of certain conformal field theories (CFTs) can be used as trial wavefunctions, and further that the wavefunctions would then inherit the non-trivial topological properties of the CFT [2, 4].

Perhaps the most interesting of the quantum Hall states that have been constructed using CFT is the Read–Rezayi series [5] some of which are actually thought to exist in nature [2]. These wavefunctions can be described as the densest polynomial wavefunctions that satisfy a particular clustering condition—that the wavefunction does not vanish when k particles come to the same point, but does vanish when the $k+1^{st}$ particle arrives (this simple rule describes the $\mathbb{Z}_k$ Read–Rezayi wavefunction for bosons, a more complicated rule describes the analog for fermions). Because of the success of the Read–Rezayi wavefunctions, generalizations of this clustering rule are worth considering. Although the rule could be generalized in many different ways, one approach has recently been proposed that seems particularly interesting [6–8]. In this approach, quantum Hall wavefunctions are described as being so-called Jack symmetric functions [9] (or 'Jacks'). The mathematical structure of the Jacks allows a detailed study of these wavefunctions, and the Jacks include the Read–Rezayi wavefunctions, as well as other previously proposed wavefunctions [10–12], as special cases. Interestingly, the fact that these Jacks obey a generalized clustering rule was previously pointed out in the mathematical literature [13, 14], and in that work it was conjectured that these Jack polynomials should be describable as correlators of certain $\mathcal{W}$-algebra CFTs. This correspondence was proven rigorously in a special case [14] (the $k = 2$ case, which corresponds to the Virasoro minimal model CFTs $M(3, 2 + r)$ in notation described below). However, for the general case, the connection remains a conjecture. One purpose of this paper is to add substantially to the evidence for this correspondence. We do this by devising a rather general method that can be used to extract the central charge from a wavefunction that exhibits a particular ($\mathbb{Z}_k$-like) clustering property. The central charge comes out as a coefficient deeply embedded in the ground-state wavefunction. When used on Jacks described by unitary models, our method gives a central charge that equals the one obtained through the fundamentally different method of counting edge excitations in [8]. For non-unitary theories, the edge method and the method derived in this paper result in different values of the central charge. The results of the edge method [8] correspond to the so-called 'effective central charge' of the $\mathcal{W}$-algebra whereas the method presented in this paper directly obtains the central charge of the same theory. We also show how to obtain the fundamental quasihole scaling exponent as a coefficient embedded in the *un-normalized* quasihole wavefunction obtained in [8]. When used on Jacks described by unitary models, the scaling dimension appears consistent with that previously obtained through the computation of edge correlators on the disk in [8] (they do not appear consistent for non-unitary models). Although we apply it only to Jack polynomials, our method works for any k-clustered wavefunction. The second purpose of this paper is to examine some of the basic properties of the $\mathcal{W}$-algebras and their applicability as fractional quantum Hall wavefunctions.

The outline of this paper is as follows. In section 2, we will briefly review the connection between quantum Hall wavefunctions and conformal field theories and present the strategy for obtaining the central charge from a generic k-clustered wavefunction. In section 3.1, we introduce the Jack polynomial wavefunctions. We assume that the Jack polynomials are described by a CFT with certain properties and we derive several properties of this putative CFT—including the central charge in section 3.2 and scaling dimensions of certain quasihole fields in section 3.3. In section 4.1, we introduce the general $\mathcal{W}$-algebra CFTs (in particular the $\mathcal{W}$-algebras based on $sl(k)$ or $\mathcal{W}_k$ algebras). In section 4.2, we discuss the construction of quantum Hall wavefunctions using these CFTs. In section 4.3, we show that the CFTs $\mathcal{W}_k(k+1, k+r)$, sometimes notated $\mathcal{WA}_{k-1}(k+1, k+r)$, precisely match the derived properties of the Jacks. Rather interestingly we find that, with the exception of the Read–Rezayi series (including the Moore–Read state and the Laughlin states), all of the Jacks correspond to non-unitary CFTs. General arguments, presented in a series of recent papers by Read [24, 25],

appear to preclude such wavefunctions from representing a gapped phase of matter, although they could correspond to critical points between phases. Gapless excitations have not yet been identified and the precise meaning of the non-unitarity is still under investigation. A recent manuscript [15] has proposed a method by which unitary, albeit Abelian, theories may be built from non-unitary ones. This work builds on the observation [16, 17] that the Abelian Jain state is a 2-quasielectron–2-quasihole excitation of the non-unitary Gaffnian state (which allows a Jack polynomial description).

We note that other $\mathcal{W}$-algebras exist which are unitary, although many other CFTs also exist with similar clustering properties. We point out that other $\mathcal{W}$-algebra wavefunctions, which would be unitary, are also possible.

2. Constructing quantum Hall states using conformal field theories

In this section, we review the construction of quantum Hall states from CFTs. A more detailed discussion is given in [2, 4, 5]. Those familiar with this topic may be able to skip much of this section.

We will consider CFTs with a simple current ψ_1 having $\mathbb{Z}_k$ symmetry (i.e., it fuses with itself k times to give the identity). In the original work on parafermions [26], the operator product expansion (OPE) for such a theory is given generally by the following (with $n = 0, \ldots, k-1$ being defined modulo k):

$$\lim_{z \to z'} \psi_n(z)\psi_{n'}(z') \sim (z - z')^{\Delta_{nn'}} \psi_{(n+n')\mathrm{mod}k}(z') + \cdots, \tag{1}$$

where ψ_0 is interpreted as the identity field I and '$\cdots$' indicates less singular terms, and

$$\Delta_{nn'} = h_{[(n+n')\mathrm{mod}k]} - h_n - h_{n'}. \tag{2}$$

Here h_n is the conformal dimension (scaling dimension or conformal weight) of the field ψ_n which we assume to be given by the expression

$$h_n = \frac{rn(k-n)}{2k} \tag{3}$$

with $r \geqslant 2$ being an integer. The usual parafermions of Fateev and Zamolodchikov [26] are recovered for $r = 2$. For other values of $r > 2$, we obtain a modified parafermion-like theory. Indeed, such a modification was proposed very briefly in appendix A of [26]. We will refer to a CFT of this form as being $(r/2)th$ generation $\mathbb{Z}_k$ CFT, and we will use the notation $\mathbb{Z}_k^{(r/2)}$ proposed by [27]. Note that all of the cases of this type that we are aware of with r odd correspond to non-unitary theories. (The fact that odd r are allowed was apparently first pointed out in [28].) Note that generically, this OPE, without specification of further terms in the expansion, is not sufficient to completely define the CFT, and there may be many allowable CFTs that fit this description of $\mathbb{Z}_k^{(r/2)}$. These different possible theories are distinguished, among other ways, by their central charges.

We will further assume that in the relevant CFT, there are no additional conserved currents. In this case should any two primary fields fuse to give the identity I, conformal invariance gives us [3]

$$\lim_{z \to z'} \phi(z)\phi'(z') = (z - z')^{-2h}[I + (z - z')^2(2h/c)T(z') + \cdots], \tag{4}$$

where h is the scaling dimension (conformal weight) of the primary fields ϕ and ϕ' (these dimensions are necessarily equal if they fuse to I), c is the central charge of the theory and T is the stress–energy tensor that satisfies the OPE [3]

$$\lim_{z \to z'} T(z)\phi(z') = \frac{h}{(z - z')^2}\phi(z') + \cdots \tag{5}$$

for any primary field ϕ with scaling dimension h.

Given a conformal field theory with these properties, we can construct the multiparticle wavefunction as a correlator [2, 4, 5]

$$\Psi(z_1, \ldots, z_N) = \langle \psi_1(z_1)\psi_1(z_2)\cdots\psi_1(z_N)\rangle \prod_{i<j}(z_i - z_j)^{\frac{r}{k}+M} \tag{6}$$

with M being a non-negative integer (throughout most of this paper we will assume $M = 0$). We will assume that the number N of particles is a multiple of k (otherwise the correlator term is strictly zero). Note that the usual Gaussian factors that occur for wavefunctions in the lowest Landau level are not written explicitly here (see [2, 4] for further discussion of this issue). The case of even M will correspond to a boson wavefunction and odd M will correspond to a fermion (electron) wavefunction.

The fusion relation equation (1) gives us

$$\lim_{z_1\to z_2} \psi_1(z_1)\psi_1(z_2) \sim (z_1 - z_2)^{-\frac{r}{k}}\psi_2(z_2) + \cdots. \tag{7}$$

This fractional power is precisely canceled by the fractional Jastrow factor in equation (6) so that the wavefunction is properly single valued in the electron coordinate.

It will sometimes be convenient to think of the above Jastrow factors in equation (6) as having resulted from vertex operators $e^{i\beta\varphi(z)}$ for φ being a free massless scalar Bose field satisfying

$$\langle \varphi(z_1)\varphi(z_2)\rangle = -\log(z_1 - z_2) \tag{8}$$

such that we have the operator product expansion

$$e^{ia\varphi(z_1)}\, e^{ib\varphi(z_2)} \sim (z_1 - z_2)^{ab} \tag{9}$$

which results in the conformal weight [3] (scaling dimension) of $e^{i\varphi\beta}$ being $\beta^2/2$. Strictly speaking, the correlator of these vertex operators is zero unless a neutrality condition is satisfied. This issue is ignored as we assume a smeared background charge (this background charge also reintroduces the above neglected Gaussian factors [4]).

Now we can define the 'electron' operator

$$\psi_e(z) = \psi_1(z)\, e^{i\varphi(z)\beta} \tag{10}$$

and choosing

$$\beta = \sqrt{M + r/k} \tag{11}$$

we can rewrite equation (6) as

$$\Psi(z_1, \ldots, z_N) = \langle \psi_e(z_1)\psi_e(z_2)\cdots\psi_e(z_N)\rangle. \tag{12}$$

Again for M even this is a fully symmetric wavefunction and for M odd a fully antisymmetric one. By using the OPEs, it is easy to establish that in the $M = 0$ case the wavefunction does not vanish as k particles come to the same point, but vanishes as r powers when the $k + 1$ st particle arrives: this is a simple k-cluster wavefunction in the notation of [29]. As noted in that work, such wavefunctions do not exist for kr odd, and correspondingly no $\mathbb{Z}_k^{(r/2)}$ theory exists for kr odd. For general M, the wavefunction vanishes as $n(n-1)M/2$ powers as $n \leqslant k$ particles come together and vanishes as $(k+1)kM/2 + r$ powers when the $k + 1$th arrives.

In the case of M even, the wavefunction is fully symmetric corresponding to a wavefunction for bosons, and we should expect the elementary 'electron' field ψ_e to have an integer dimension. However, with M even the scaling dimension of ψ_e (the sum of the dimensions of ψ_1 and the vertex) is an integer only for even r and is a half-integer for r odd. Conversely for M odd, one has a fully antisymmetric wavefunction, but ψ_e has a half-integer dimension only for r even. This should make one suspect that there are some problems for the case of odd r,

and indeed there are no unitary theories for the case of odd r which precludes the possibility of odd r wavefunctions representing gapped phases of matter. (It would be nice to develop a deeper understanding of precisely how these two facts are related.)

Using the arguments of [2, 4, 5, 29], it is easy to establish that the degree N_ϕ of the polynomial wavefunction Ψ is given by

$$N_\phi = \nu^{-1} N - \mathcal{S}, \tag{13}$$

where

$$\nu = \frac{k}{r + kM} \tag{14}$$

is the filling fraction and

$$\mathcal{S} = r + M \tag{15}$$

is the so-called 'Shift' on the sphere. From here on, we will be considering quantum Hall effect of bosons and we will consider the case of $M = 0$ for simplicity. Generalization to other values of M is relatively trivial.

The chosen CFT will typically contain many other primary field operators. Suppose the conformal field theory contains a field σ with operator product expansion

$$\lim_{z_1 \to z_2} \sigma(z_1)\psi_1(z_2) \sim (z_1 - z_2)^{\Delta_{\sigma 1}} \phi(z_2) + \cdots. \tag{16}$$

Here we must have

$$\Delta_{\sigma 1} = h_\phi - h_1 - h_\sigma, \tag{17}$$

where h_σ and h_ϕ are the conformal weights (scaling dimensions) of the fields σ and ϕ, respectively. The fact that there is only one conformal family on the right-hand side of equation (16) is guaranteed by the assumption that ψ_1 is a simple current.

We then define a quasihole operator

$$\psi^\sigma_{qh}(z) = \sigma(z)\, \mathrm{e}^{\mathrm{i}\varphi(z)\gamma}, \tag{18}$$

where

$$\gamma = (s - \Delta_{\sigma 1})/\beta \tag{19}$$

with s being a non-negative integer. This choice of γ is the only possibility that will make the wavefunction Ψ (equation (20)) properly single valued in the electron coordinates z_i. The resulting wavefunction can be written out as

$$\Psi(z_1, \ldots, z_N; w_1, \ldots, w_n) = \langle \psi_{qh}(w_1) \cdots \psi_{qh}(w_n) \psi_e(z_1) \cdots \psi_e(z_N) \rangle \tag{20}$$

$$= \langle \sigma(w_1) \cdots \sigma(w_n) \psi_1(z_1) \cdots \psi_1(z_N) \rangle \times \prod_{i<j} (z_i - z_j)^{\beta^2} \prod_{k=1}^{N} \prod_{m=1}^{n} (z_k - w_m)^{s - \Delta_{\sigma 1}} \prod_{p<m}^{n} (w_p - w_m)^{\gamma^2}. \tag{21}$$

The charge on such a quasihole is given by the exponent $s - \Delta_{\sigma 1}$ which pushes away a corresponding fraction of the ambient density from the position of each quasihole. It is easy to show [5] that the resulting charge must be

$$e^*_{qh} = (s - \Delta_{\sigma 1})\, e\nu, \tag{22}$$

where $-e$ is the charge of the electron. Note that, for example, by fusing with p electron fields we can also create quasiparticles with change $e^*_{qh} - pe$.

One trivial possibility is to choose the field σ to be the identity field (with dimension 0). In this case, the identity field fuses with ψ_1 to give ψ_1 again (so $\phi = \psi_1$ in equation (16)) and $\Delta_{\sigma 1} = 0$. The quasihole is then given by $s = 1$ in (the $s = 0$ case gives the identity operator). We identify this case as the Laughlin quasihole and we then write

$$\psi_{qh}^{\text{Laughlin}}(z) = \mathrm{e}^{\mathrm{i}\varphi(z)/\beta}. \tag{23}$$

This can be inserted into a wavefunction resulting in a Jastrow factor. For example,

$$\left\langle \psi_{qh}^{\text{Laughlin}}(w)\psi_e(z_1)\cdots\psi_e(z_N)\right\rangle = \langle\psi_e(z_1)\cdots\psi_e(z_N)\rangle \prod_{k=1}^{N}(w - z_N), \tag{24}$$

where the correlator on the right-hand side is the wavefunction in the absence of quasiholes. The charge of this quasihole is $e_{qh}^* = e\nu$ as is expected for a quasihole created by a single flux insertion [1].

More generally, however, for non-trivial fields σ the charge on the quasihole will be some fraction of the Laughlin value. Furthermore, when several non-trivial (non-Laughlin) quasiholes are created, the correlator separates into conformal blocks. This is the hallmark of non-Abelian statistics—the fact that there are several orthogonal wavefunctions that describe the set of quasiholes at one particular set of positions [2, 4]. By examining the fusion rules of the quasihole operators σ, we can count the number of conformal blocks and determine the degeneracy associated with the non-Abelian statistics.

2.1. A strategy for obtaining the central charge of a theory from wavefunctions

Given a polynomial wavefunction in the lowest Landau level, we would like to identify a CFT which gives this wavefunction as a correlator. Let us imagine that we are given a bosonic polynomial wavefunction $\Psi(z_1, z_2, z_3, z_4, \ldots, z_N)$ with several properties that make it compatible with CFTs of parafermion type as described above. We assume that the wavefunction does not vanish as k particles come together, but vanishes as r powers when the $k + 1$ st arrives (we assume $M = 0$ bosons in the language above). We further assume that it is a simple cluster wavefunction [29], meaning that it is filling fraction $\nu = k/r$ with shift r. As described above, such wavefunctions are compatible with CFTs of parafermion type, and in particular can be compatible with $\mathbb{Z}_k^{(r/2)}$ CFTs.

We extract the putative correlator by removing the Jastrow factor (see equation (6))

$$\phi_0(z_1, z_2, \ldots, z_N) = \frac{\Psi(z_1, z_2, \ldots, z_N)}{\prod_{i<j}^{N}(z_i - z_j)^{1/\nu}}. \tag{25}$$

We will further assume that the CFT we are searching for is rational—that the number of primary fields is finite. Our general strategy will be to successively fuse k coordinates together to obtain the identity field again (see equation (1)). The subleading term of the final fusion (see equation (4)) allows us to produce the stress–energy tensor T with a coefficient that depends on the central charge.

Now let $z_1 = z_2 + \epsilon_1$ and let $\epsilon_1 \to 0$; then expand in ϵ_1. Given the expected OPE equation (1), we should obtain something of the form

$$\begin{aligned}\phi_0(z_2 + \epsilon_1, z_2, \ldots, z_N) = {} & \frac{1}{\epsilon_1^{2h_1 - h_2}}(\phi_1(z_2, \ldots, z_N) \\ & + \epsilon_1\phi_{1,1}(z_2, \ldots, z_N) + \epsilon_1^2\phi_{1,2}(z_2, \ldots, z_N) + \cdots).\end{aligned} \tag{26}$$

Since the function Ψ is given to us, we can easily determine $\phi_1, \phi_{1,1}, \ldots$ explicitly. If the divergence exponent is not $2h_1 - h_2$ then we conclude that the CFT (if it exists) is not of the $\mathbb{Z}_k^{(r/2)}$ parafermion type. In terms of correlators, the function ϕ_1 should be given by

$$\phi_1 = \langle \psi_2(z_2)\psi_1(z_3)\psi_1(z_4)\cdots\psi_1(z_N)\rangle. \tag{27}$$

Now assuming that $k > 2$ we repeat the procedure taking $z_2 = z_3 + \epsilon_2$ and let $\epsilon_2 \to 0$; then expand in ϵ_2 to obtain

$$\begin{aligned}\phi_1(z_3+\epsilon_2, z_3, \ldots, z_N) = {} & \frac{1}{\epsilon_2^{h_1+h_2-h_3}}(\phi_2(z_3, \ldots, z_N) \\ & + \epsilon_2\phi_{2,1}(z_3, \ldots, z_N) + \epsilon_2^2\phi_{2,2}(z_3, \ldots, z_N) + \cdots).\end{aligned} \tag{28}$$

We continue this procedure $k-2$ times. We finally obtain

$$\phi_{k-2} = \langle \psi_{k-1}(z_{k-1})\psi_1(z_k)\psi_1(z_{k+1})\cdots\psi_1(z_N)\rangle. \tag{29}$$

Taking the last limit we obtain

$$\begin{aligned}\phi_{k-2}(z_k+\epsilon_{k-1}, z_k, \ldots, z_N) = {} & \frac{1}{\epsilon_{k-1}^{2h_1}}(\phi_{k-1}(z_k, z_{k+1}, \ldots, z_N) \\ & + \epsilon_{k-1}^2\phi_{k-1,2}(z_k, z_{k+1}, \ldots, z_N) + \cdots).\end{aligned} \tag{30}$$

Here we have used the OPE equation (4),

$$\phi_{k-1}(z_k, z_{k+1}, \ldots, z_N) = \langle I(z_k)\psi_1(z_{k+1})\cdots\psi_1(z_N)\rangle = \langle \psi_1(z_{k+1})\cdots\psi_1(z_N)\rangle, \tag{31}$$

which should be independent of the position z_k. Note that there is no term $\phi_{k-1,1}$ linear in ϵ_{k-1} in the expansion (again, if this is not true, it is evident that we do not have a CFT of parafermion type). Indeed, it will be useful below to note that the subleading term vanishes when the leading term is the identity. The second term, on the other hand, from equation (4), gives us

$$\phi_{k-1,2}(z_k, z_{k+1}, \ldots, z_N) = (2h_1/c)\langle T(z_k)\psi_1(z_{k+1})\cdots\psi_1(z_N)\rangle. \tag{32}$$

We now take one more limit setting $z_k = z_{k+1} + \epsilon_k$ and let $\epsilon_k \to 0$; then expand in ϵ_k. Using the OPE equation (5) we obtain

$$\phi_{k-1,2}(z_{k+1}+\epsilon, z_{k+1}, \ldots, z_N) = \frac{1}{\epsilon_k^2}\left[\phi_k(z_{k+1}, \ldots, z_N) + \cdots\right], \tag{33}$$

where

$$\phi_k = \left(2h_1^2/c\right)\langle \psi_1(z_{k+1})\cdots\psi_1(z_N)\rangle. \tag{34}$$

Thus by taking the ratio

$$\frac{\phi_k(z_k, \ldots, z_N)}{\phi_{k-1}(z_k, \ldots, z_N)} = 2h_1^2/c, \tag{35}$$

we are able to extract the putative central charge. A similar scheme will also be used below to extract scaling dimensions of quasiparticles.

3. Jack wavefunctions

3.1. The basics of using Jack symmetric functions as quantum Hall wavefunctions

In this section, we describe the construction of quantum Hall wavefunctions as Jack symmetric functions. This reviews work of [6–8, 13, 14, 20]. In addition to the interesting clustering

properties of the Jacks mentioned in section 1, the bosonic Jacks as well as their fermionic counterparts (equal to bosonic Jacks times Vandermonde determinant) have special properties [20, 35] that make them simple to work with both analytically and numerically. Notably, they can be evaluated numerically far easier than by usual techniques of exact diagonalization of generalized Haldane pseudopotentials [18] currently used to obtain model wavefunctions. Methods developed in [35] allow for a factor of 1000 improvement in computation time for model wavefunctions over prior methods. Expressing other wavefunctions in terms of their Jack components can similarly result in large numerical speedups.

The Jack symmetric functions (Jacks) are polynomials satisfying a number of particular properties. We refer the reader to [9, 20–23] for a more detailed discussion of many of these properties. We write a general Jack as

$$J_\lambda^\alpha(z_1, z_2, \ldots, z_N). \tag{36}$$

This is a function of N complex variables z_i and parametrically depends on a real so-called 'Jack-parameter' α, as well as a partition λ of length $|\lambda|$ where $|\lambda| \leqslant N$. A partition λ is an ordered set of numbers $\lambda_i \leqslant \lambda_{i-1}$, $1 \leqslant i \leqslant |\lambda|$ such that

$$\sum_{i=1}^{|\lambda|} \lambda_i = \ell_\lambda, \tag{37}$$

where ℓ_λ is some integer number. Each partition can be uniquely associated with a Young diagram [22] in the standard way. Note that we have followed the usual convention that a partition is made up of positive integers with no integer equal to zero. However, frequently below we will want to think of the partition λ as having exactly N pieces, thus we can do this by including in addition $N - |\lambda|$ occurrences of the integer 0.

A detailed definition of the Jack polynomial is given in appendix A for the interested reader. For the present, however, it suffices to state that the Jacks are simply polynomials satisfying a great number of interesting properties that have been previously worked out [9, 20–23].

In a recent work by Haldane and Bernevig [6–8] it was pointed out that setting the Jack parameter

$$\alpha = -(k+1)/(r-1), \tag{38}$$

with $k+1$ and $r-1$ coprime, generates symmetric polynomials (which we think of as bosonic quantum Hall wavefunctions, corresponding to $M = 0$ above) satisfying the admissibility condition that the wavefunction vanishes as r powers when $k+1$ particles come to the same point. This admissibility condition had been noted previously in the mathematical literature [13, 14] and for the Read–Rezayi states by Haldane [19]. In [6], it was shown that the requirement of translational invariance uniquely selects *all* the Jacks that can be good FQH wavefunctions, as we will discuss further below. Translational invariance immediately gives the Jack parameter $\alpha = -(k+1)/(r-1)$ as well as the (k, r) admissibility on partitions. Among the wavefunctions that can be described as Jacks are the Read–Rezayi series [5] (including the Moore–Read state [4]), the Laughlin wavefunctions [1] and the Gaffnian [12]. We emphasize that the Jack parameters here are negative rational, in contrast with other applications to condensed matter systems such as the Calogero model [45], which have positive Jack parameter.

An important ingredient of Jack polynomials is the so-called root state. This can be constructed out of the partition λ according to

$$R_\lambda = \mathcal{S}\left[\prod_{i=1}^{N} z_i^{\lambda_i}\right] \Big/ \mathcal{N}, \tag{39}$$

where the symbol $\mathcal{S}$ represents the symmetrization over all the permutations of z_i (equivalently, one may think of it as computing the permanent perm$\left(z_i^{\lambda_j}\right)$) and $\mathcal{N}$ is the normalization factor which we define in (40). It is convenient to think of a root state as representing occupation of orbitals. Imagining the orbitals $\varphi_m \sim z^m$ in the lowest Landau level in the plane, we describe the root state R_λ as a set of occupation numbers $n_m(\lambda)$ for bosons occupying orbitals where summing the total number of particles in all orbitals gives the total number of particles N. In other words, $n_m(\lambda)$ is the number of times the integer m occurs in the partition λ. For example, a root polynomial corresponding to the partition $\lambda_1 = 2, \lambda_2 = 0$, given by $R_\lambda = z_1^2 + z_2^2$, can be described in terms of occupation numbers $n_0 = 1, n_1 = 0, n_2 = 1$. In this paper, we will almost always write partitions in terms of occupation numbers. For example, for the root polynomial introduced in this paragraph we will write $\lambda = [1, 0, 1]$, where the terms in the square brackets are $n_m(\lambda)$ for $m = 0, 1, 2$. (See [6, 7] for more details on the translation between the orbital occupation representation of partitions and the conventional representation of partitions.) In terms of $n_m(\lambda)$, the normalization factor $\mathcal{N}$ is given by

$$\mathcal{N} = \prod_m n_m!. \tag{40}$$

Its role is simply to eliminate any additional factors which might arise from symmetrizing already symmetric expressions.

A Jack symmetric polynomial is not simply equal to its root state. Each Jack is labeled by a root state, but is actually a superposition of the root state along with many other descendant states which can be constructed by 'squeezing' occupation numbers—i.e., which can be obtained by starting with the root state and moving bosons toward each other in pairs [6, 7]. In the language of partitions, the descendent states are *dominated* by the root, or highest weight, state. The Jack is given by a particular combination of the root state and its descendants which make it an eigenvalue of a differential operator known as the Laplace–Beltrami operator [20].

Not every Jack polynomial can correspond to a quantum Hall wavefunction. Indeed, some of them are not even translationally invariant (that is, they change under the change of variables $z_i \to z_i + a$). In [6], it was shown that the Jacks that correspond to translationally invariant wavefunctions are (a) those with Jack parameter as in equation (38) and (b) have partitions corresponding to root states with the property that no more than k bosons may occupy r consecutive orbitals, for some given k and r. It is interesting that in the limit of a thin cylinder, the root state is precisely the wavefunction, which means that the entire physics just becomes an issue of distributing bosons so as to satisfy the admissibility condition. This fact has been exploited in a number of recent publications [30–34].

Given N coordinates, with N divisible by k, the root partition that yields the lowest degree polynomial (and hence the highest density wavefunction) is given by the occupation numbers

$$\lambda = [k \underbrace{00\cdots0}_{r-1 \text{ times}} k \underbrace{00\cdots0}_{r-1 \text{ times}} k \cdots \underbrace{00\cdots0}_{r-1 \text{ times}} k], \tag{41}$$

where there are N/k orbitals filled with k bosons each. We abbreviate this occupation with the obvious notation

$$\lambda = [k0^{r-1}k0^{r-1}k\cdots0^{r-1}k]. \tag{42}$$

If we consider the corresponding Jack symmetric function to be a wavefunction for bosons

$$\Psi = J^{\alpha}_{k0^{r-1}k0^{r-1}k\ldots0^{r-1}k}(z_1, \ldots, z_N) \tag{43}$$

with α as above, we generate a wavefunction that vanishes as r powers when $k+1$ coordinates approach each other, but does not vanish when k coordinates come to the same point. As discussed in [6, 7], this is a wavefunction at filling fraction $\nu = k/r$ with shift $\mathcal{S} = r$. Thus,

this is a simple k-cluster wavefunction as discussed in [29]. Further, this suggests that such a wavefunction may be described as a correlator of a $\mathbb{Z}_k^{(r/2)}$ CFT as described in the previous section (with $M = 0$). We will show additional evidence below that this is indeed the case.

One can similarly describe quasihole states in terms of Jacks that have lower density root partitions. For example, if we allow the N bosons to occupy one additional orbital, admissible root partitions include

$$[0k0^{r-1}k0^{r-1}k\cdots 0^{r-1}k] \tag{44}$$

$$[1(k-1)0^{r-1}k0^{r-1}k\cdots 0^{r-1}k] \tag{45}$$

$$[k0^{r}k0^{r-1}k\cdots 0^{r-1}k] \tag{46}$$

$$[k0^{r-1}(k-2)20^{r-1}k\cdots 0^{r-1}k] \tag{47}$$

$$\vdots$$

and many others. These many possibilities correspond to both the multiple types of quasiparticles and the many positions where the quasiparticles may be placed. One may analyze all of the possibilities to categorize the possible quasiparticle types, and further one can consider how these quasiparticle types fuse with each other. This exercise has been performed [33, 34] in the context of the thin cylinder limit, and it has been found that these admissibility rules correspond to the particles and fusion rules of $su(k)_r$. However, one must be cautious that many bulk states with differing properties may have the same thin torus limit. Nonetheless, since the Jacks are completely described by their thin limit (which is just their root state), this calculation immediately implies that the fusion relations of quasiholes of the (k, r) admissible Jacks are precisely that of the $su(k)_r$.

3.2. The central charge of the Jack polynomials

We now extract the central charge for the Jack polynomial wavefunctions. We consider a wavefunction corresponding to the (k, r) Jack as in equation (43), fixing α as in equation (38) throughout (we do not write the parameter α explicitly from here on).

We now proceed as in section 2.1 to extract the central charge. As above, we begin with

$$\phi_0 = \langle \psi_1(z_1)\psi_1(z_2)\cdots\psi_1(z_N)\rangle = \frac{J_{k0^{r-1}k0^{r-1}k\ldots k0^{r-1}k}}{\prod_{i<j=1}^{N}(z_i - z_j)^{\frac{1}{\nu}}}. \tag{48}$$

Without loss of generality, since the polynomials are translationally invariant [6], we may choose $z_1 = 0$. This is quite convenient as the only orbital which does not vanish as $z \to 0$ is the $m = 0$ orbital, so it is easy to see when certain wavefunctions do or do not vanish. Further, the remaining Jack polynomial, having taken this limit, is given by the partition $[(k-1)0^{r-1}k0^{r-1}k\cdots k0^{r-1}k]$ with a coefficient of unity since all our Jack polynomials share the 'monic' normalization.

Now take the limit $z_2 \to 0$, and expand as in equation (26). For $k > 2$, there will be a term proportional to z_2 in the expansion (as in equation (26)), which means that we have not fused to the identity, and we simply take the leading term of the expansion, which is given by the Jack with partition $[(k-2)0^{r-1}k0^{r-1}k\cdots k0^{r-1}k]$. In the process, from the divergence of the Jastrow factor we find the relation

$$h_1 + h_1 - h_2 = \frac{1}{\nu}. \tag{49}$$

We now take the coordinate z_3 to zero, and so forth, until we reach $J_{10^{r-1}k0^{r-1}k\cdots k0^{r-1}k}$. During this process, keeping track of successive divergencies of the Jastrow factor we find the series of equations for the scaling dimensions which are

$$h_1 + h_m - h_{m+1} = \frac{m}{\nu}, \tag{50}$$

where $m = 1, \ldots, k-1$ (with $h_k = 0$), which is consistent with our expectation from equation (3).

We now focus on the last fusion of this series (see equation (30)) from which we will get the central charge. Here, we let $z_k \to 0$:

$$\begin{aligned}\frac{J_{10^{r-1}k0^{r-1}k\cdots k0^{r-1}k}(z_k, z_{k+1}, \ldots, z_N)}{\prod_{i=k}^N z_i^{\frac{k-1}{\nu}} \prod_{i<j=k}^N (z_i - z_j)^{\frac{1}{\nu}}} &= \frac{1}{z_k^{\frac{k-1}{\nu}} \prod_{i=k+1}^N z_i^{\frac{k}{\nu}} \prod_{i<j=k+1}^N (z_i - z_j)^{\frac{1}{\nu}}} \\ &\times \left(J_{0^r k0^{r-1}k\cdots k0^{r-1}k}(z_{k+1}, \ldots, z_N) + z_k P_1 + z_k^2 P_2 + \cdots\right) \\ &\times \left(1 + z_k \frac{1}{\nu} \sum_{j=k+1}^N \frac{1}{z_j} + z_k^2 \left(\frac{1}{2}\frac{1}{\nu}\left(1 + \frac{1}{\nu}\right) \sum_{j=k+1}^N \frac{1}{z_j^2} + \frac{1}{\nu^2} \sum_{i<j=k+1}^N \frac{1}{z_i z_j}\right) + \cdots\right).\end{aligned} \tag{51}$$

The polynomials P_1, P_2 can be obtained by expanding the Jack polynomial $J_{10^{r-1}k0^{r-1}k\cdots k0^{r-1}k}(z_k, z_{k+1}, \ldots, z_N)$ for small z_k. In this expansion, the resulting polynomials are in fact other Jacks with the same value of α, i.e., they are (k, r) admissible. This expansion is shown explicitly in appendix B giving

$$\begin{aligned}J_{10^{r-1}k0^{r-1}k\cdots k0^{r-1}k}(z_k, z_{k+1}, \ldots, z_N) =& J_{0^r k0^{r-1}k\cdots k0^{r-1}k}(z_{k+1}, \ldots, z_N) \\ &+ z_k A_1 J_{0^{r-1}1(k-1)0^{r-1}k\cdots k0^{r-1}k}(z_{k+1}, \ldots, z_N) \\ &+ z_k^2 [B_1 J_{0^{r-2}10(k-1)0^{r-1}k\cdots k0^{r-1}k}(z_{k+1}, \ldots, z_N) \\ &+ B_2 J_{0^{r-1}1(k-1)0^{r-2}1(k-1)0^{r-1}k\cdots k0^{r-1}k}(z_{k+1}, \ldots, z_N)] + \cdots.\end{aligned} \tag{52}$$

For simplicity of notation, let us define the following notation:

$$\begin{aligned}P_0 &= J_{0^r k0^{r-1}k\cdots k0^{r-1}k}(z_{k+1}, \ldots, z_N) \\ P_{1,0} &= J_{0^{r-1}1(k-1)0^{r-1}k\cdots k0^{r-1}k}(z_{k+1}, \ldots, z_N) \\ P_{2,0} &= J_{0^{r-2}10(k-1)0^{r-1}k\cdots k0^{r-1}k}(z_{k+1}, \ldots, z_N) \\ P_{2,1} &= J_{0^{r-1}1(k-1)0^{r-2}1(k-1)0^{r-1}k\cdots k0^{r-1}k}(z_{k+1}, \ldots, z_N),\end{aligned} \tag{53}$$

where, of course, in equation (51) we have $P_1 = A_1 P_{1,0}$ and $P_2 = B_1 P_{2,0} + B_2 P_{2,1}$. The coefficients A_1, B_1, B_2 are to be determined. Of these coefficients, A_1, B_1 are simple to evaluate whereas B_2 is somewhat harder. Fortunately, we will not actually need to fully evaluate B_2. Leaving the values of these coefficients unspecified for the moment (we derive them in appendix B), we have, for the products of brackets in equation (51),

$$\begin{aligned}&\frac{1}{z_k^{\frac{k-1}{\nu}} \prod_{i=k+1}^N z_i^{\frac{k}{\nu}} \prod_{i<j=k+1}^N (z_i - z_j)^{\frac{1}{\nu}}} \left(J_{0^r k0^{r-1}k\cdots k0^{r-1}k}(z_{k+1}, \ldots, z_N) + z_k P_1 + z_k^2 P_2 + \cdots\right) \\ &\quad\times \left(1 + z_k \frac{1}{\nu} \sum_{j=k+1}^N \frac{1}{z_j} + z_k^2 \left(\frac{1}{2}\frac{1}{\nu}\left(1 + \frac{1}{\nu}\right) \sum_{j=k+1}^N \frac{1}{z_j^2} + \frac{1}{\nu^2} \sum_{i<j=k+1}^N \frac{1}{z_i z_j}\right) + \cdots\right) \\ &= \frac{1}{z_k^{\frac{k-1}{\nu}} \prod_{i=k+1}^N z_i^{\frac{k}{\nu}} \prod_{i<j=k+1}^N (z_i - z_j)^{\frac{1}{\nu}}} \left(P_0 + z_k \left(A_1 P_{1,0} + \frac{1}{\nu} \sum_{j=k+1}^N \frac{1}{z_j} \cdot P_0\right)\right.\end{aligned}$$

$$+ z_k^2 \left(\left(\frac{1}{2}\frac{1}{\nu}\left(1+\frac{1}{\nu}\right)\sum_{j=k+1}^{N}\frac{1}{z_j^2} + \frac{1}{\nu^2}\sum_{i<j=k+1}^{N}\frac{1}{z_i z_j}\right)\cdot P_0 \right.$$
$$\left. + A_1 P_{1,0}\cdot\frac{1}{\nu}\sum_{j=k+1}^{N}\frac{1}{z_j} + B_1 P_{2,0} + B_2 P_{2,1}\right) + \cdots \Bigg) \tag{54}$$

The coefficient A_1 is derived in appendix B and is given by $A_1 = -\frac{1}{\nu}$. We also have, as shown in appendix E, the identity

$$\sum_i \frac{\partial}{\partial z_i} P_0 = r P_{1,0}. \tag{55}$$

We can thus derive a series of identities

$$\begin{aligned}\sum_i \frac{\partial}{\partial z_i} J_{0^r k 0^{r-1} k\cdots k 0^{r-1} k}(z_{k+1},\ldots,z_N) &= r J_{0^{r-1}1(k-1)0^{r-1}k\cdots k0^{r-1}k}(z_{k+1},\ldots,z_N)\\ &= \sum_i \frac{\partial}{\partial z_i}\prod_j z_j^r J_{k0^{r-1}k\cdots k0^{r-1}k}(z_{k+1},\ldots,z_N)\\ &= r\sum_i \frac{1}{z_i}\prod_j z_j^r J_{k0^{r-1}k\cdots k0^{r-1}k}(z_{k+1},\ldots,z_N)\\ &= r\sum_i \frac{1}{z_i} J_{0^r k 0^{r-1}k\cdots k0^{r-1}k},\end{aligned} \tag{56}$$

where we have used the fact that $J_{k0^{r-1}k\cdots k0^{r-1}k}(z_{k+1},\ldots,z_N)$ is a highest weight state by virtue of being a ground state. The above shows that

$$P_{1,0} = \sum_i \frac{1}{z_i} P_0. \tag{57}$$

Then we have

$$\left(A_1 P_{1,0} + \frac{1}{\nu}\sum_{j=k+1}^{N}\frac{1}{z_j}\cdot P_0\right) = \frac{1}{\nu}\left(-P_{1,0} + \sum_{j=k+1}^{N}\frac{1}{z_j}\cdot P_0\right) = 0 \tag{58}$$

so the first-order term in z_k vanishes as it should. We now have

$$\begin{aligned}&\frac{1}{\prod_{i=k+1}^{N} z_i^{\frac{k}{\nu}}\prod_{i<j=k+1}^{N}(z_i-z_j)^{\frac{1}{\nu}}}\left(J_{0^r k 0^{r-1}k\cdots k0^{r-1}k}(z_{k+1},\ldots,z_N) + z_k P_1 + z_k^2 P_2\right)\\ &\qquad\times\left(1 + z_k\frac{1}{\nu}\sum_{j=k+1}^{N}\frac{1}{z_j} + z_k^2\left(\frac{1}{2}\frac{1}{\nu}\left(1+\frac{1}{\nu}\right)\sum_{j=k+1}^{N}\frac{1}{z_j^2} + \frac{1}{\nu^2}\sum_{i<j=k+1}^{N}\frac{1}{z_i z_j}\right)\right)\\ &= \frac{1}{\prod_{i=k+1}^{N} z_i^{\frac{k}{\nu}}\prod_{i<j=k+1}^{N}(z_i-z_j)^{\frac{1}{\nu}}}\left(P_0 + z_k^2\left(\left(\frac{1}{2}\frac{1}{\nu}\left(1-\frac{1}{\nu}\right)\sum_{j=k+1}^{N}\frac{1}{z_j^2}\right.\right.\right.\\ &\qquad\left.\left.\left. - \frac{1}{\nu^2}\sum_{i<j=k+1}^{N}\frac{1}{z_i z_j}\right)\cdot P_0 + B_1 P_{2,0} + B_2 P_{2,1}\right) + \cdots\right)\end{aligned} \tag{59}$$

Note that $k/\nu = r$. A few additional identities, shown in appendix E, are now also useful

$$\frac{P_0}{\prod_{i=k+1} z_i^{k/\nu}} = J_{k0^{r-1}k\cdots k0^{r-1}k}(z_{k+1}, \ldots, z_N) \tag{60}$$

$$\frac{P_{2,0}}{\prod_{i=k+1} z_i^{k/\nu}} = \frac{1}{\prod_{i=k+1} z_i^2} J_{10(k-1)0^{r-1}k\cdots k0^{r-1}k}(z_{k+1}, \ldots, z_N) \tag{61}$$

$$\frac{P_{2,1}}{\prod_{i=k+1} z_i^{k/\nu}} = \frac{1}{\prod_{i=k+1} z_i} J_{1(k-1)0^{r-2}1(k-1)\cdots k0^{r-1}k}(z_{k+1}, \ldots, z_N). \tag{62}$$

Now as in equation (33) let $z_{k+1} \to 0$, and isolate the singularities in $1/z_{k+1}^2$. We only need the leading singularity, so we can immediately see that $P_{2,1}$ does not matter because it is less singular (after being multiplied by $\prod_{i=k+1} z_i^{-k/\nu}$). Also when $z_{k+1} \to 0$ we have, to leading order,

$$\begin{aligned}
&\frac{1}{\prod_{i=k+1} z_i^{k/\nu}} \left(\frac{1}{2}\frac{1}{\nu}\left(1-\frac{1}{\nu}\right) \sum_{j=k+1}^{N} \frac{1}{z_j^2} - \frac{1}{\nu^2} \sum_{i<j=k+1}^{N} \frac{1}{z_i z_j} \right) \cdot P_0 \\
&\quad = \frac{1}{2}\frac{1}{\nu}(1-\frac{1}{\nu})\frac{1}{z_{k+1}^2} J_{k0^{r-1}k\cdots k0^{r-1}k}(z_{k+1}=0, \ldots, z_N) \\
&\quad = \frac{1}{2}\frac{1}{\nu}\left(1-\frac{1}{\nu}\right)\frac{1}{z_{k+1}^2} J_{(k-1)0^{r-1}k\cdots k0^{r-1}k}(z_{k+2}, \ldots, z_N)
\end{aligned} \tag{63}$$

also

$$\begin{aligned}
\frac{P_{2,0}}{\prod_{i=k+1} z_i^{k/\nu}} &= \frac{1}{\prod_{i=k+1} z_i^2} J_{10(k-1)0^{r-1}k\cdots k0^{r-1}k}(z_{k+1}, \ldots, z_N) \\
&\underbrace{=}_{z_{k+1}\to 0} \frac{1}{z_{k+1}^2}\frac{1}{\prod_{i=k+2} z_i^2} J_{10(k-1)0^{r-1}k\cdots k0^{r-1}k}(z_{k+1}=0, \ldots, z_N) \\
&= \frac{1}{z_{k+1}^2}\frac{1}{\prod_{i=k+2} z_i^2} J_{00(k-1)0^{r-1}k\cdots k0^{r-1}k}(z_{k+2}, \ldots, z_N) \\
&= \frac{1}{z_{k+1}^2} J_{(k-1)0^{r-1}k\cdots k0^{r-1}k}(z_{k+2}, \ldots, z_N).
\end{aligned} \tag{64}$$

We are now in a position to obtain the final equation: as $z_{k+1} \to 0$ we have

$$\begin{aligned}
&\frac{1}{\prod_{i=k+1}^{N} z_i^{\frac{k}{\nu}}} \left(P_0 + z_k^2 \left(\left(\frac{1}{2}\frac{1}{\nu}\left(1-\frac{1}{\nu}\right) \sum_{j=k+1}^{N} \frac{1}{z_j^2} - \frac{1}{\nu^2} \sum_{i<j=k+1}^{N} \frac{1}{z_i z_j} \right) \cdot P_0 + B_1 P_{2,0} + B_2 P_{2,1} \right) \right) \\
&\quad = J_{k0^{r-1}k\cdots k0^{r-1}k}(z_{k+1}, \ldots, z_N) + \frac{1}{z_{k+1}^2}\left(\frac{1}{2}\frac{1}{\nu}\left(1-\frac{1}{\nu}\right) + B_1 \right) \\
&\qquad \times J_{k-10^{r-1}k\cdots k0^{r-1}k}(z_{k+2}, \ldots, z_N).
\end{aligned} \tag{65}$$

The coefficient B_1 is derived in appendix B giving

$$B_1 = \frac{r(r-1)}{2}\frac{\alpha+1}{((r-1)\alpha+1)((r-2)\alpha+1)} = -\frac{r}{2k}\frac{(r-k-2)(r-1)}{-kr+2k+1}. \tag{66}$$

Taking the appropriate ratio as in equation (35), we then have

$$\frac{2h_1^2}{c} = \frac{1}{2}\frac{1}{\nu}\left(1-\frac{1}{\nu}\right) + B_1, \tag{67}$$

where $h_1 = \frac{r(k-1)}{2k}$. With a trivial algebra we obtain

$$c = r\frac{k-1}{k+r}(2k+1-kr). \tag{68}$$

This is precisely the central charge of the $\mathcal{W}_k(k+1, k+r)$ CFT [38–40] as will be discussed further below.

3.3. The quasihole scaling exponent for Jack polynomials

In this section, we obtain the quasihole scaling dimension from the polynomial wavefunctions. The strategy is quite similar to that described above—fusion of fields together to form the stress–energy tensor as in equation (4) giving a prefactor proportional to the field dimension.

We must first write a wavefunction that describes an elementary quasihole at some position w and another object with which it can uniquely fuse to form the identity at the origin. This other object is precisely the fusion of $k-1$ quasiholes. These wavefunctions can be established by invoking the clustering conditions as described in [8]:

$$\Psi(w; z_1, \ldots, z_N)|_{z_1=z_2\cdots=z_k=w} = 0 \tag{69}$$

$$\Psi(w; z_1, \ldots, z_N)|_{z_1=z_2=0} = 0. \tag{70}$$

The unique such admissible wavefunction is a superposition of the Jacks given by [8]

$$\Psi(w; z_1, \ldots, z_N) = \sum_{i=0}^{N/k} \left(-\frac{w}{k}\right)^i |i\rangle, \tag{71}$$

where $|i\rangle$ are the Jack polynomials

$$|0\rangle = J_{0k0^{r-1}k0^{r-1}k\cdots 0^{r-1}k0^{r-1}k} \tag{72}$$

$$|1\rangle = J_{1(k-1)0^{r-1}k0^{r-1}k\cdots 0^{r-1}k0^{r-1}k} \tag{73}$$

$$|2\rangle = J_{1(k-1)0^{r-2}(k-1)0^{r-1}k\cdots 0^{r-1}k0^{r-1}k} \tag{74}$$

$$\vdots \tag{75}$$

and the Jack parameter α is as above always taken to be $-(k+1)/(r-1)$.

If we had generated this wavefunction from a conformal field theory (see equation (20)), there would generically be an additional non-single-valued dependence on the quasihole coordinate w (i.e., there may be branch cuts). However, this really just provides a normalization for the wavefunction whereas the Jacks are not normalized with respect to integration over all the z coordinates (and supposedly the CFT correlator is normalized, see [24, 36, 37] for a detailed discussion of this issue). We thus expect that there may be some arbitrary prefactor $f(w, 0)$ which multiplies our Jack wavefunction in order to produce the CFT wavefunction.

We thus write a proposed correlator corresponding to a single quasihole at w and $k-1$ quasiholes at the origin:

$$\langle\sigma'(0)\sigma(w)\psi_e(z_1)\psi_e(z_2)\cdots\psi_e(z_N)\rangle = f(w,0)\frac{\Psi(w; z_1, \ldots, z_N)}{\prod_{i=1}^{N}(z_i-0)^{\frac{k-1}{k}}(z_i-w)^{\frac{1}{k}}}. \tag{76}$$

Here we have notated the primary field associated with the quasihole as σ and the field associated with the fusion of $k-1$ quasiholes as σ'. These two fields fuse to give the identity

as in equation (4). We now want to let the quasihole coordinate $w \to 0$ and examine the result of this fusion. Keeping terms up to order w^2 we obtain

$$\langle \sigma'(0)\sigma(w)\psi_e(z_1)\cdots\psi_e(z_N)\rangle = \frac{f(w,0)}{\prod_{i=1}^N z_i}\left(|0\rangle - \frac{w}{k}|1\rangle + \left(\frac{w}{k}\right)^2|2\rangle + \cdots\right)$$
$$\times\left(1 + \frac{1}{k}w\sum_{i=0}^N\frac{1}{z_i} + w^2\left(\frac{1}{2}\frac{1}{k}\left(1+\frac{1}{k}\right)\sum_{i=0}^N\frac{1}{z_i^2} + \frac{1}{k^2}\sum_{i<j}\frac{1}{z_i z_j}\right) + \cdots\right). \quad (77)$$

We now massage the terms in the product of the two brackets. The term independent of w is $|0\rangle$. Thus, the unknown prefactor $f(w,0)$ must contain the divergent prefactor shown in equation (4). The term linear in w is

$$\frac{1}{k}\left(-|1\rangle + \sum_{i=0}^N\frac{1}{z_i}|0\rangle\right). \quad (78)$$

Considering the form of equation (4) we now must show that this linear term vanishes. This is demonstrated explicitly in appendix D. We now go to the second-order terms and write them as

$$\left(\frac{1}{2}\frac{1}{k}\left(1+\frac{1}{k}\right)\sum_{i=0}^N\frac{1}{z_i^2} + \frac{1}{k^2}\sum_{i<j}\frac{1}{z_i z_j}\right)|0\rangle - \frac{1}{k^2}\sum_{i=0}^N\frac{1}{z_i}|1\rangle + \frac{1}{k^2}|2\rangle$$
$$= \left(\frac{1}{2}\frac{1}{k}\left(1-\frac{1}{k}\right)\sum_{i=0}^N\frac{1}{z_i^2} - \frac{1}{k^2}\sum_{i<j}\frac{1}{z_i z_j}\right)|0\rangle + \frac{1}{k^2}|2\rangle, \quad (79)$$

where we have again used the identity $\sum_i (1/z_i)|0\rangle = |1\rangle$ proved in appendix D. So we have hence proved that for $w \to 0$ we have

$$\langle \sigma'(0)\sigma(w)\psi(z_1)\cdots\psi(z_N)\rangle = \frac{f(w,0)}{\prod_{i=1}^N z_i}$$
$$\times\left(|0\rangle + w^2\left(\left(\frac{1}{2}\frac{1}{k}\left(1-\frac{1}{k}\right)\sum_{i=0}^N\frac{1}{z_i^2} - \frac{1}{k^2}\sum_{i<j}\frac{1}{z_i z_j}\right)|0\rangle + \frac{1}{k^2}|2\rangle\right) + \cdots\right). \quad (80)$$

To make this look nicer, we can write (again using the same identity) $|0\rangle = J_{0k0^{r-1}k0^{r-1}k\cdots k0^{r-1}k} = \prod_{i=1}^N z_i J_{k0^{r-1}k0^{r-1}k\cdots k0^{r-1}k} = \prod_{i=1}^N z_i \Psi_{GS}$ where $\Psi_{\rm GS}$ is the ground state in the absence of the quasiholes (equation (12)). Hence we have

$$\langle \sigma'(0)\sigma(w)\psi_e(z_1)\cdots\psi_e(z_N)\rangle = f(w,0)$$
$$\times\left(\Psi_{\rm GS} + w^2\left(\left(\frac{1}{2}\frac{1}{k}\left(1-\frac{1}{k}\right)\sum_{i=0}^N\frac{1}{z_i^2} - \frac{1}{k^2}\sum_{i<j}\frac{1}{z_i z_j}\right)\Psi_{\rm GS} + \frac{1}{k^2}\frac{1}{\prod_{i=1}^N z_i}|2\rangle\right) + \cdots\right). \quad (81)$$

Thus from equation (4) we have

$$(2h_\sigma/c)\langle T(0)\psi_e(z_1)\cdots\psi_e(z_N)\rangle$$
$$= \left(\frac{1}{2}\frac{1}{k}\left(1-\frac{1}{k}\right)\sum_{i=0}^N\frac{1}{z_i^2} - \frac{1}{k^2}\sum_{i<j}\frac{1}{z_i z_j}\right)\Psi_{\rm GS} + \frac{1}{k^2}\frac{1}{\prod_{i=1}^N z_i}|2\rangle. \quad (82)$$

Note that T is the stress–energy tensor of the CFT not including the bose vertex operators. We now let one of the electron coordinates approach the position 0 giving a leading singularity

$$\lim_{z_1\to 0}(2h_\sigma/c)\langle T(0)\psi_e(z_1)\cdots\psi_e(z_N)\rangle = \frac{1}{z_1^2}\frac{1}{2}\frac{1}{k}\left(1-\frac{1}{k}\right)\Psi_{GS}+\cdots. \tag{83}$$

From the OPE equation (5) we thus conclude that

$$2\frac{h_\sigma h_1}{c} = \frac{1}{2}\frac{k-1}{k^2}. \tag{84}$$

Using the above expression (equation (3)) for h_1 we then obtain

$$h_\sigma = \frac{c}{2kr}, \tag{85}$$

thus relating the scaling dimension of the quasiparticle to the central charge. As expected this result gives precisely the scaling dimension of the minimal quasiparticle field of the $\mathcal{W}_k(k+1,k+r)$ CFT [38–40] as will be discussed further below.

In fact, since we also know that the charge of the quasiparticle is $-e\nu/k$ (there is a k-fold fractionalization of the Laughlin quasiparticle), then by equations (17) and (22) we know immediately that the fusion of the quasiparticle with the electron field must create another field ϕ with scaling dimension

$$h_\phi = h_1 + h_\sigma + 1/k. \tag{86}$$

Indeed, again in the $\mathcal{W}_k(k+1,k+r)$ CFT, the fusion of σ with ψ_1 creates a field of precisely this scaling dimension, as we will see below.

In principle, with enough effort, one can calculate the scaling dimension of other fields using similar techniques.

4. $\mathcal{W}$-algebra quantum Hall wavefunctions

In this section, we describe a very large family of CFTs known as $\mathcal{W}$-algebras. As discussed above, these CFTs may be used in construction of quantum Hall wavefunctions, and further, some of them apparently correspond to the wavefunctions obtained from Jack polynomials.

4.1. Introductory facts about some $\mathcal{W}$-algebras

A great deal is known about $\mathcal{W}$-algebras, and the variety of $\mathcal{W}$-algebras that have been studied in the literature is both vast and growing. We refer the reader to [38] for general information about this field. In this paper, we will focus on some of the simplest cases known which are the minimal models of the $\mathcal{W}_k$ algebras. These minimal models were first described in [39, 41]. In this paper, we will follow the approaches of [39, 40], and then specify (and simplify) to the situations in which we are interested (there are, however, other ways to describe the same CFTs; see [38]). Nothing in this section is new, but is rather just a distillation of prior results from these references.

Each of the simple Lie algebra $\mathcal{A}_{n-1}, \mathcal{B}_n, \mathcal{D}_n, \mathcal{E}_6, \mathcal{E}_7, \mathcal{E}_8$ can be associated with a $\mathcal{W}$-algebra. Among these we will only be interested in $\mathcal{WA}_{k-1}$ which has $\mathbb{Z}_k$ symmetry. These are sometimes notated as just $\mathcal{W}_k$. We leave for future research the question of what quantum Hall states can be constructed from other $\mathcal{W}$-algebras [38].

Focusing on the algebra $\mathcal{W}_k$, a field $\Phi(\mathbf{l};\mathbf{l}')$ is specified by two vectors $\mathbf{l}$ and $\mathbf{l}'$ of $k-1$ positive integers

$$(\mathbf{l};\mathbf{l}') = (l_1,\ldots,l_{k-1};l_1',\ldots,l_{k-1}'). \tag{87}$$

The fusion of such fields can be written symbolically as

$$\Phi(\mathbf{l};\mathbf{l}')\times\Phi(\mathbf{m};\mathbf{m}')=\sum_{(\mathbf{s};\mathbf{s}')}[\Phi(\mathbf{s};\mathbf{s}')]. \tag{88}$$

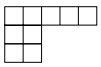
The values of $(\mathbf{s};\mathbf{s}')$ that contribute to the right-hand side for a given $(\mathbf{l};\mathbf{l}')$ and $(\mathbf{m};\mathbf{m}')$ are known as the 'qualitative structure' or 'selection rules' of the algebra. Treating $(\mathbf{l};\mathbf{l}')$ as the corresponding specified representation of $sl(k)\times sl(k)$ gives the proper structure. Equivalently, we think of the vectors

$$(l_1-1,l_2-1,\ldots,l_{k-1}-1) \tag{89}$$

$$(l'_{k-1}-1,\ldots,l'_2-1,l'_1-1) \tag{90}$$

as the Dynkin labels of two $SU(k)$ representations [3] so that $(\mathbf{l};\mathbf{l}')$ are labels for a representation of $SU(k)\times SU(k)$. The qualitative structure of the fusion in $SU(k)\times SU(k)$ then is the qualitative structure of our $\mathcal{W}$-algebra. It is useful to note that when all of the l_i values are unity, we have the trivial, or identity, representation.

To relate these labels to more familiar notation, we note that Dynkin labels can be trivially converted to Young tableau [3]. In $SU(k)$, the $k-1$ labels give the respective differences between the number of boxes in each of the $k-1$ successive rows. So for example, for $k=5$ the Dynkin labels $(3,0,2,0)$ represent the tableau

 . (91)

The identity representation is the empty tableau (although for clarity we will sometimes write **1**). Note that to write a field configuration, we must increase each Dynkin label by 1 and note that the primed indices are read right to left and the unprimed left to right. Thus, a typical field for the case of $k=5$ can be written graphically as

$$\Phi(4,1,3,1;2,2,2,1)=\left(\;;\;\right). \tag{92}$$

A minimal model has parameters tuned so that a finite algebra closes. In particular, we specify a minimal model by two integers p and p' both greater than k which are relatively prime. This theory is typically called $\mathcal{W}_k(p,p')$ or $\mathcal{WA}_{k-1}(p,p')$. The fields $\Phi(\mathbf{l};\mathbf{l}')$ that form a closed algebra are given by the set satisfying the constraint

$$\sum_{i=1}^{k-1}l_i\leqslant p'-1 \tag{93}$$

$$\sum_{i=1}^{k-1}l'_i\leqslant p-1. \tag{94}$$

These constraints can be thought of as restricting the corresponding Young tableaus to have no more than $p'-k$ and $p-k$ columns, respectively. This type of restriction in the number of columns of a tableau is analogous to what happens when one looks at representations of $SU(k)$ at level m where m is $p'-k$ or $p-k$ respectively.

Furthermore, in the minimal model there is an association of fields

$$\Phi(\mathbf{l};\mathbf{l}')=\Phi(\tilde{\mathbf{l}}_m;\tilde{\mathbf{l}}'_m)\qquad m=1,\ldots,(k-1), \tag{95}$$

where

$$\tilde{\mathbf{l}}_m = (l_{k-m+1}, \ldots, l_{k-1}, l_0, l_1, \ldots, l_{k-m-1}) \tag{96}$$

$$\tilde{\mathbf{l}}'_m = (l'_{k-m+1}, \ldots, l'_{k-1}, l'_0, l'_1, \ldots, l'_{k-m-1}) \tag{97}$$

with

$$l_0 = p' - \sum_{i=1}^{k-1} l_i \tag{98}$$

$$l'_0 = p - \sum_{i=1}^{k-1} l'_i. \tag{99}$$

The number of Young tableaus in $SU(k)$ with no more than m columns (and as usual no more than $k-1$ rows) is given by $\binom{k-1}{m}$. Thus, we can apparently write $(p-1)!(p'-1)!/((p-k)!(p'-k)!(k-1)!^2)$ fields Φ, specified by a combination of two different Young tableaus (one with no more than $p-k$ rows and one with no more than $p'-k$ rows). However, the k-fold identification of fields in equation (95) leaves us with only

$$\frac{(p-1)!(p'-1)!}{(p-k)!(p'-k)!(k-1)!k!} \tag{100}$$

distinct primary fields in this algebra.

It is convenient to define[5]

$$\alpha_\pm = \frac{\pm 1}{\sqrt{2}}(p/p')^{\pm\frac{1}{2}} \tag{101}$$

$$\alpha_0 = \alpha_+ + \alpha_- = (p-p')/\sqrt{2pp'}. \tag{102}$$

The central charge of this algebra is

$$c_k(p;p') = (k-1)\left(1 - 2k(k+1)\alpha_0^2\right) \tag{103}$$

$$= (k-1)\left(1 - \frac{k(k+1)(p-p')^2}{pp'}\right). \tag{104}$$

The algebra is unitary [42] if and only if $p = p' \pm 1$, which is the only case where the central charge can be positive. In a dynamical theory, the central charge has the interpretation of a density of states or heat capacity.

The scaling dimension (or conformal weight) of a field $\Phi(\mathbf{l};\mathbf{l}')$ is given by

$$h(\mathbf{l};\mathbf{l}') = -\alpha_0^2 k(k^2-1)/12 + \left[\sum_{i=1}^{k-1}\sum_{j=1}^{k-1}(l_i\alpha_+ + l'_i\alpha_-)F_{ij}(l_j\alpha_+ + l'_j\alpha_-)\right] \tag{105}$$

where F is a symmetric $k-1$ dimensional matrix F with matrix elements

$$F_{ij} = j(k-i)/k \qquad j \leqslant i. \tag{106}$$

(This is the inverse of the Cartan matrix of $SU(k)$.)

[5] We follow the convention of [39] for these variables. The definitions of [40] differ by factors of 2.

This theory has a conserved $\mathbb{Z}_k$ charge. The charge of a field $\Phi(\mathbf{l};\mathbf{l}')$ is given by

$$q = \left(\sum_{n=1}^{k-1}[mn(l_n - 1) - m'n(l'_n - 1)]\right) \bmod k, \tag{107}$$

where $m = p \bmod k$ and $m' = p' \bmod k$.

Finally, we note that for non-unitary theories, an interesting quantity to define is the so-called 'effective central charge' given by

$$\tilde{c} = c - 24h_{\min}, \tag{108}$$

where $h_{\min}$ is the conformal weight of the primary field with the smallest (most negative) dimension (sometimes known as the pseudovacuum field). This quantity is necessarily positive, and like the central charge in unitary theories, represents a density of levels. For the minimal $\mathcal{W}_k(p, p')$ theories being considered in this section, we have [43]

$$\tilde{c}_k(p, p') = (k-1)\left(1 - \frac{k(k+1)}{pp'}\right). \tag{109}$$

4.2. Cluster wavefunctions from $\mathcal{W}$-algebras

Examining the structure of this $\mathcal{W}$-algebra, we note that there always exist simple currents corresponding to the desired parafermion fields from equation (1). Given p and p' relatively prime and greater than k, we examine the minimal model $\mathcal{W}_k(p, p')$. In this algebra, we find simple currents

$$\psi_1 = \Phi(p' - k + 1, 1, 1, \ldots, 1; 1, 1, \ldots, 1) \tag{110}$$

$$\psi_2 = \Phi(1, p' - k + 1, 1, \ldots, 1; 1, 1, \ldots, 1) \tag{111}$$

$$\vdots$$

$$\psi_{k-1} = \Phi(1, 1, \ldots, 1, p' - k + 1; 1, 1, \ldots, 1) \tag{112}$$

and of course the identity operator is given by

$$\psi_0 = \Phi(1, 1, 1, \ldots, 1; 1, 1, \ldots, 1). \tag{113}$$

It is worth noting that due to the field identification equation (95), these parafermion fields can equally well be expressed as

$$\psi_1 = \Phi(1, 1, \ldots, 1; 1, 1, \ldots, 1, p - k + 1) \tag{114}$$

$$\psi_2 = \Phi(1, 1, \ldots, 1; 1, 1, \ldots, p - k + 1, 1) \tag{115}$$

$$\vdots$$

$$\psi_{k-1} = \Phi(1, 1, \ldots, 1; p - k + 1, 1, \ldots, 1). \tag{116}$$

In terms of the Young tableau, we can express these fields compactly (in both representations) as

$$\psi_n = \left(\begin{array}{c} \overbrace{}^{p'-k \text{ columns}} \\ \left.\begin{array}{c}\square\square\square\square\\ \square\square\square\square\end{array}\right\} n \text{ rows} \end{array} ; \mathbf{1} \right) \tag{117}$$

$$= \left(\mathbf{1} ; \begin{array}{c} \overbrace{}^{p-k \text{ columns}} \\ \left.\begin{array}{c}\square\square\square\square\\ \square\square\square\square\end{array}\right\} n \text{ rows} \end{array} \right), \tag{118}$$

where we have written **1** for the identity representation or the empty tableau.

Using equation (105) we determine that the scaling dimension of these parafermion fields is indeed given by the desired equation (3) with

$$r = (p-k)(p'-k). \tag{119}$$

Hence $\mathcal{W}_k(p, p')$ is a CFT of the $\mathbb{Z}_k^{(r/2)}$ type as described above. Using equation (6), we find that the algebra $\mathcal{W}_k(p, p')$ constructs a quantum Hall wavefunction at filling fraction and shift (see equation (14))

$$\nu = \frac{k}{(p-k)(p'-k) - kM} \qquad \mathcal{S} = (p-k)(p'-k) + M \tag{120}$$

with M being a non-negative integer. For $M = 0$, this wavefunction has the property that it is a simple k-cluster wavefunction, that is, it does not vanish when k particles come to the same point, but it vanishes as r powers when the $k+1$st arrives. We emphasize that for arbitrary r, there are generically many CFTs that can correspondingly generate many inequivalent wavefunctions with this property (it appears, however, that at least for $r = 2$ and $k = 2$, $r = 3$, the CFT is uniquely defined by this property). Note that the unitary CFTs correspond to $p = p' \pm 1$. Thus we expect a series of unitary wavefunctions at filling fraction and shift

$$\nu = \frac{k}{m(m+1) - kM} \qquad \mathcal{S} = m(m+1) + M \tag{121}$$

with $m > k$. Note again that $m = 2$ is the Read–Rezayi series. The case of $m = 3$ and $k = 2$ corresponds to the tricritical Ising CFT, which has previously been proposed for a quantum Hall wavefunction by [24, 44].

4.3. The series $\mathcal{W}_k(k+1, k+r)$

The $\mathcal{W}$-algebras of interest corresponding to the above-discussed Jack polynomials are $\mathcal{W}_k(k+1, k+r)$. Plugging $p = k+1$ and $p' = k+r$ into the above expression gives us a wavefunction with the properties described in the previous section. Here, we must choose k and r both integers $\geqslant 2$ and where $k+1$ and $k+r$ are relatively prime. As discussed above, these algebras have $\mathbb{Z}_k$ symmetry and central charge (plugging into equation (104))

$$c = \frac{(k-1)(1-k(r-2))r}{k+r} \tag{122}$$

which matches the central charge of the (k, r) Jack polynomial found in equation (68).

The case of $r = 2$ is the $\mathbb{Z}_k$ parafermion [26] CFT model which describes the corresponding $\mathbb{Z}_k$ Read–Rezayi wavefunctions [5]. It is easy to see that the $r = 2$ case is the only value of r for which the central charge is positive, and is hence the only case where $\mathcal{W}_k(k+1, k+r)$ is unitary. The case of $k = 2$ simplifies to precisely the Virasoro minimal

model [3] $\mathcal{M}(3, 2+r)$. The $k = 2, r = 2$ case here, which is the Ising conformal field theory, corresponds to the Moore–Read Pfaffian [4]. The $k = 2, r = 3$ case corresponds to the recently discussed Gaffnian wavefunction [12].

Using the fact that $p = k+1$ here, because of the identification of fields mentioned above in equation (95), all fields may be written as a single Young tableau and the identity

$$\Phi = (\text{tableau}; \mathbf{1}), \tag{123}$$

where the possible tableaus are in a one-to-one correspondence with those of $su(k)_r$. Furthermore, it turns out that fusion rules of these fields are also precisely those of $su(k)_r$. This connection was noted in [34] where, as mentioned above, the fusion rules of (k, r) cluster states in the thin torus limit were also found to be that of $su(k)_r$. Thus we point to this as further evidence of the connection between $\mathcal{W}_k(k+1, k+r)$ and the (k, r) Jack wavefunctions.

We now examine some of the fields of this $\mathcal{W}$-algebra in detail. In addition to the ψ_n fields described above, we now examine a possible quasiparticle field. Let us consider a field

$$\sigma = (\square; \mathbf{1}) \tag{124}$$

whose $\mathbb{Z}_k$ charge is 1 and scaling dimension is (see equations (107) and (105))

$$h_\sigma = \frac{(k-1)(1+k(2-r))}{2k(r+k)}. \tag{125}$$

It is easy to check that this indeed satisfies the predicted relationship equation (85). The fusion of the simple current with this elementary spin field is particularly simple. We have

$$\psi_1 \times \sigma = \phi, \tag{126}$$

where ϕ is the field

$$\phi = \left(\begin{array}{c} r \text{ columns} \\ \ydiagram{4,1} \end{array} ; \mathbf{1} \right) \tag{127}$$

which has scaling dimension (again using equation (105))

$$h_\phi = \frac{2k^2 - (k+r)^2 + k(r^2-3)}{2k(r+k)}. \tag{128}$$

In equation (16) we have

$$\Delta_{\sigma 1} = h_\phi - h_\sigma - h_1 = -1/k \tag{129}$$

as predicted by equation (86). Thus, the quantum Hall state generated by this CFT has a quasiparticle of charge $e^* = -e\nu/k$. It is easy to check that this is the lowest charge quasiparticle that can be constructed from the theory.

From equation (109), we calculate the effective central charge $\tilde{c}$ which we find to be given by

$$\tilde{c} = \frac{r(k-1)}{k+r}. \tag{130}$$

Interestingly this is the value of the central charge found in [8] by counting the density of edge modes on a disk for (k, r) Jack wavefunctions. (Strictly speaking, this reference finds $\tilde{c}+1$, where the +1 corresponds to the $U(1)$ charge boson.) Thus, it appears to be the effective central charge that determines the density of states. This is perhaps not surprising since the central charge of the non-unitary $\mathcal{W}$-models is negative, and a negative density of states would

be unphysical. Also in [8], the scaling dimension of the quasiparticle is bounded numerically. While the result of this calculation agrees with the $\mathcal{W}$-model prediction for unitary cases, it does not agree for the non-unitary cases. This apparently contradictory result is not currently understood.

5. Summary

We have shown here that the (k, r) Jack quantum Hall states likely correspond to the $\mathcal{W}_k(k+1, k+r)$ CFTs. We have shown that both have simple currents ψ_n with $\mathbb{Z}_k$ symmetry having the scaling dimension given by equation (3). They both have an elementary quasiparticle field σ having the scaling dimension given by equation (85), and ψ_1 fuses with σ to yield another field ϕ with the scaling dimension given by equation (86). And, as shown previously in [34], the fusion algebras of both systems are identical. While this does not completely prove that the two theories are equivalent, it is very strong evidence. We comment that for the case of $k = 2$, a full proof of equivalence has been given by [14].

Making connection to prior work of [8] we find that the central charge determined in that work by edge state analysis in that work agrees with the $\mathcal{W}$-algebra central charge in the unitary cases and corresponds to the effective central charge of the $\mathcal{W}$-algebra in the non-unitary cases (see also [46] where the effective central charge edge physics was found for the Haldane–Rezayi state). However, the analysis of the quasiparticle exponent in that work does not appear to be in agreement with the current $\mathcal{W}$-algebra analysis in non-unitary cases.

It is interesting that among all of the wavefunctions described by the Jack polynomials, only the Read–Rezayi series and the Laughlin series correspond to unitary CFTs. Presumably this means [24, 25] that, other than these specific cases, the Jack polynomials cannot correspond to gapped phases of matter. Nonetheless, they can still correspond to critical points between phases, and understanding the nature of this criticality can teach us much about the adjacent phases. Other unitary $\mathcal{W}$-algebras could in principle correspond to gapped phases. We note however, as mentioned in [24], that the identification of a CFT for use as a quantum Hall wavefunction does not yet imply that this wavefunction is the ground state of a Hamiltonian. Further work will be required to try to construct such Hamiltonians [18] for any proposed wavefunction.

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Appendix A. Detailed definition of the Jack polynomials

There are several equivalent ways to define the Jack polynomials [9, 20–23]. The reader should note, however, that there are several standard inequivalent normalizations that are used. Note also that in this appendix, the standard description of a partition is used rather than the occupation basis description.

Let us write the form of the Jack polynomial from [21] (see also [20]). We start by defining the Jacks which are a function of a single variable

$$J_k^\alpha(z_1) = z_1^k(1+\alpha)\cdots(1+(k-1)\alpha). \tag{A.1}$$

We then define each Jack in terms of Jacks with one fewer variables

$$J^{\alpha}_{\lambda}(z_1, z_2, \ldots, z_N) = \sum_{\mu \subseteq \lambda} J^{\alpha}_{\mu}(z_2, \ldots, z_N) z_1^{|\lambda/\mu|} \beta_{\lambda\mu}, \tag{A.2}$$

the summation is over all subpartitions μ of λ ($\mu \subseteq \lambda$) such that the skew partition λ/μ is a so-called horizontal strip. Here, λ/μ is a horizontal strip if $\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \geqslant \lambda_3 \geqslant \mu_3 \ldots$, or in other words, if no two distinct points of λ/μ, regarded as the difference $\lambda - \mu$ of their Young diagrams, lie in the same column (draw the diagram of λ then color only the squares of λ that do not belong to μ; if no two colored squares are in the same column then λ/μ is a horizontal strip). The exponent $|\lambda/\mu|$ in the above equation is just the difference $\sum_i \lambda_i - \sum_j \mu_j$ (note that λ and μ do not have to have identical length).

The coefficients $\beta_{\lambda,\mu}$ are given by (see proposition 6.1 of [21])

$$\beta_{\lambda,\mu} = \frac{\prod_{(i,j)\in\lambda} B^{\lambda}_{\lambda,\mu}(i, j)}{\prod_{(i,j)\in\mu} B^{\mu}_{\lambda,\mu}(i, j)}, \tag{A.3}$$

where

$$B^{\nu}_{\lambda,\mu}(i, j) = h^{*}_{\nu}(i, j) \qquad \text{if} \quad \lambda'_j = \mu'_j; = h^{\nu}_{*}(i, j) \text{ otherwise.} \tag{A.4}$$

In the above, λ' and μ' are the conjugate partitions of λ, μ, obtained by transposing the Young diagram of the partition λ, μ (i.e., one writes them as Young diagrams and then interchanges the rows with the columns). h^{*}_{λ} and h^{λ}_{*} are the generalized upper and lower hook lengths of the partition λ:

$$h^{*}_{\lambda}(i, j) = \lambda'_j - i + \alpha(\lambda_i - j + 1), \qquad h^{\lambda}_{*}(i, j) = \lambda'_j - i + 1 + \alpha(\lambda_i - j), \tag{A.5}$$

the product $\prod_{(i,j)\in\lambda}$ means product over *all* the pairs (i, j) in the Young tableau of partition λ, i.e., i goes over all the components of the partition $\lambda = (\lambda_i)$ whereas, for a set i, j runs from 1 to λ_i.

Unfortunately, this relatively simple definition is not the normalization that we use within this paper. The above normalization corresponds to the normalization of Stanley [20], in which the coefficient of the root monomial of the Jack polynomial J^{α}_{λ} is equal to $v_{\lambda\lambda}(\alpha)$, where

$$v_{\lambda\lambda}(\alpha) = \prod_{(i,j)\in\lambda} h^{\lambda}_{*}(ij). \tag{A.6}$$

In this paper, on the other hand, we use the 'monic' normalization where the coefficient of the 'root' (or dominating) monomial is equal to 1. Hence the coefficients $\beta_{\lambda\mu}$ in our case read

$$\beta_{\lambda,\mu} = \frac{v_{\mu\mu}(\alpha)}{v_{\lambda\lambda}(\alpha)} \frac{\prod_{(i,j)\in\lambda} B^{\lambda}_{\lambda,\mu}(i, j)}{\prod_{(i,j)\in\mu} B^{\mu}_{\lambda,\mu}(i, j)}. \tag{A.7}$$

Note that there are cancelations between v's and B's which lead to a simplified form, which we do not write explicitly.

Appendix B. Expansion of the Jack polynomials

In the main text, equation (52), we need to find the expansion of the Jack of partition $\lambda = [10^{r-1}k0^{r-1}k \cdots k0^{r-1}k]$ into Jacks of one fewer variable. Using the recursive definition of the Jack functions in the above appendix, this expansion is straightforward, and we can easily obtain the coefficients of the Jacks corresponding to the partitions

$$\mu^{A_1} = [0^{r-1}1k - 10^{r-1}k \cdots k0^{r-1}k] \tag{B.1}$$

$$\mu^{B_1} = [0^{r-2}10k - 10^{r-1}k \cdots k0^{r-1}k] \tag{B.2}$$

$$\mu_{B_2} = [0^{r-1}1k - 10^{r-2}1k - 1\cdots k0^{r-1}k]. \tag{B.3}$$

Note that since the coefficient B_2 is not needed for the calculation of the central charge, we will not derive it here since it is tedious.

B.1. Coefficient A_1

Let us now compute the coefficient $A_1 = \beta_{\lambda\mu^{A_1}}$. For the two partitions λ and μ^{A_1} we have $\lambda_i = \mu_i^{A_1}\ \forall\ i = 1,\ldots,N-k-1$ where N is the original number of particles ($N = l(\lambda) + k = l(\mu^{A_1}) + k$, where $l(\lambda)$ is the length of the partition λ—i.e., the number of elements of λ without counting the zeros—which means the length of λ is also equal to the length of μ^{A_1}. The only place where λ differs from μ^{A_1} is the last element $\mu_{N-k}^{A_1} = \lambda_{N-k} - 1 = r - 1$. For the conjugate partitions $\mu^{A_1'}$ and λ' we again have $\mu_i^{A_1'} = \lambda_i' \forall i = 1,\ldots,r-1$ and $i = r+1,\ldots,N_\Phi = \frac{r}{k}(N-k)$. The only place where they differ is $\mu_r^{A_1'} = \lambda_r' - 1 = N - k - 1$. Since all those terms are identical except for the two exceptions, in the expression of $\frac{v_{\lambda\lambda}(\alpha)}{v_{\mu\mu}(\alpha)}$ we will get cancellations except for $j = r$ and $i = 1,\ldots,N-k-1$ or for $i = N-k$ and $j = 1,\ldots,r$ (note that for μ, when $i = N - kj$ stops at $r-1$). In all other places, the partition constituents and their conjugates are identical and the ratio cancels to identity. We get

$$\frac{v_{\lambda\lambda}(\alpha)}{v_{\mu^{A_1}\mu^{A_1}}(\alpha)} = \prod_{i=1}^{N-k-1} \frac{\mu_r^{A_1'} - i + 1 + \alpha\left(\mu_i^{A_1} - r\right)}{\lambda_r' - i + 1 + \alpha(\lambda_i - r)} \prod_{j=1}^{r-1} \frac{\mu_j^{A_1'} - (N-k) + 1 + \alpha\left(\mu_{N-k}^{A_1} - j\right)}{\lambda_j' - (N-k) + 1 + \alpha(\lambda_{N-k} - j)} \times \frac{1}{\lambda_r' - (N-k) + 1 + \alpha(\lambda_{N-k} - r)} \tag{B.4}$$

Upon massaging, we get

$$\frac{1}{\lambda_r' - (N-k) + 1 + \alpha(\lambda_{N-k} - r)} = 1 \tag{B.5}$$

$$\prod_{j=1}^{r-1} \frac{\mu_j^{A_1'} - (N-k) + 1 + \alpha\left(\mu_{N-k}^{A_1} - j\right)}{\lambda_j' - (N-k) + 1 + \alpha(\lambda_{N-k} - j)} = \frac{1+\alpha(r-2)}{1+\alpha(r-1)} \frac{1+\alpha(r-3)}{1+\alpha(r-2)} \cdots \frac{1+\alpha}{1+2\alpha} \frac{1}{1+\alpha} = \frac{1}{1+\alpha(r-1)} \tag{B.6}$$

hence

$$\frac{v_{\lambda\lambda}(\alpha)}{v_{\mu^{A_1}\mu^{A_1}}(\alpha)} = \prod_{i=1}^{N-k-1} \frac{\mu_r^{A_1'} - i + 1 + \alpha\left(\mu_i^{A_1} - r\right)}{\lambda_r' - i + 1 + \alpha(\lambda_i - r)} \frac{1}{1+\alpha(r-1)}. \tag{B.7}$$

Now for B's:

$$\frac{\prod_{(i,j)\in\lambda} B^{\lambda}_{\lambda,\mu^{A_1}}(i,j)}{\prod_{(i,j)\in\mu^{A_1}} B^{\mu^{A_1}}_{\lambda,\mu^{A_1}}(i,j)} = \frac{\prod_{(i,j)\in\lambda, j\neq r} B^{\lambda}_{\lambda,\mu^{A_1}}(i,j)}{\prod_{(i,j)\in\mu^{A_1}, j\neq r} B^{\mu^{A_1}}_{\lambda,\mu^{A_1}}(i,j)} * \frac{\prod_{(i,r)\in\lambda} B^{\lambda}_{\lambda,\mu^{A_1}}(i,r)}{\prod_{(i,r)\in\mu^{A_1}, i\neq N-k} B^{\mu^{A_1}}_{\lambda,\mu^{A_1}}(i,r)}, \tag{B.8}$$

where in the last ratio, the differentiation has been made $i \neq N-k$ because $\mu_{N-k}^{A_1} = r-1$ and the point $(i,j) = (N-k, r)$ hence does not belong to the Young tableau of the partition

μ^{A_1}. We then have

$$\frac{\prod_{(i,j)\in\lambda, j\neq r} B^{\lambda}_{\lambda,\mu^{A_1}}(i,j)}{\prod_{(i,j)\in\mu^{A_1}, j\neq r} B^{\mu^{A_1}}_{\lambda,\mu^{A_1}}(i,j)} = \frac{\prod_{(i,j)\in\lambda, i\neq N-k, j\neq r} B^{\lambda}_{\lambda,\mu^{A_1}}(i,j)}{\prod_{(i,j)\in\mu^{A_1}, i\neq N-k, j\neq r} B^{\mu^{A_1}}_{\lambda,\mu^{A_1}}(i,j)} \prod_{j=1}^{r-1} \frac{B^{\lambda}_{\lambda,\mu^{A_1}}(N-k,j)}{B^{\mu^{A_1}}_{\lambda,\mu^{A_1}}(N-k,j)}$$
$$= \prod_{j=1}^{r-1} \frac{B^{\lambda}_{\lambda,\mu^{A_1}}(N-k,j)}{B^{\mu^{A_1}}_{\lambda,\mu^{A_1}}(N-k,j)}, \tag{B.9}$$

where the first product simplifies because for $i \neq N-k$ and $j \neq r$ the components of the two partitions and their conjugates are identical. Because for $j = 1, \ldots, r-1$ the components of the conjugate partitions λ'_j and $\mu^{A'_1}_j$ are equal, we have

$$\prod_{j=1}^{r-1} \frac{B^{\lambda}_{\lambda,\mu^{A_1}}(N-k,j)}{B^{\mu^{A_1}}_{\lambda,\mu^{A_1}}(N-k,j)} = \prod_{j=1}^{r-1} \frac{\lambda'_j - (N-k) + \alpha(\lambda_{N-k} - j + 1)}{\mu^{A'_1} - (N-k) + \alpha(\mu^{A_1}_{N-k} - j + 1)} = \prod_{j=1}^{r-1} \frac{r+1-j}{r-j} = r, \tag{B.10}$$

and we have solved half the products that make up the ratio of B's.

We must also refine, since $\lambda'_r \neq \mu^{A'_1}_r$:

$$\frac{\prod_{(i,r)\in\lambda} B^{\lambda}_{\lambda,\mu^{A_1}}(i,r)}{\prod_{(i,r)\in\mu^{A_1}, i\neq N-k} B^{\mu^{A_1}}_{\lambda,\mu^{A_1}}(i,r)} = \prod_{i=1}^{N-k-1} \frac{\lambda'_r - i + 1 + \alpha(\lambda_i - r)}{\mu^{A'_1}_r - i + 1 + \alpha\left(\mu^{A_1}_i - r\right)}$$
$$\times (\lambda'_r - (N-k) + 1 + \alpha(\lambda_{N-k} - r)) = \prod_{i=1}^{N-k-1} \frac{\lambda'_r - i + 1 + \alpha(\lambda_i - r)}{\mu^{A'_1}_r - i + 1 + \alpha\left(\mu^{A_1}_i - r\right)}. \tag{B.11}$$

To get to the equation:

$$\frac{\prod_{(i,j)\in\lambda} B^{\lambda}_{\lambda,\mu^{A_1}}(i,j)}{\prod_{(i,j)\in\mu^{A_1}} B^{\mu^{A_1}}_{\lambda,\mu^{A_1}}(i,j)} = r \cdot \prod_{i=1}^{N-k-1} \frac{\lambda'_r - i + 1 + \alpha(\lambda_i - r)}{\mu^{A'_1}_r - i + 1 + \alpha\left(\mu^{A_1}_i - r\right)}. \tag{B.12}$$

We finally reach the solution

$$A_1 = \beta_{\lambda\mu^{A_1}} = \prod_{i=1}^{N-k-1} \frac{\mu^{A'_1}_r - i + 1 + \alpha(\mu^{A_1}_i - r)}{\lambda'_r - i + 1 + \alpha(\lambda_i - r)} \frac{1}{1+\alpha(r-1)} r$$
$$\times \prod_{i=1}^{N-k-1} \frac{\lambda'_r - i + 1 + \alpha(\lambda_i - r)}{\mu^{A'_1}_r - i + 1 + \alpha(\mu^{A_1}_i - r)} = \frac{r}{1+\alpha(r-1)}, \tag{B.13}$$

which is $-1/v$ as mentioned in the text. Note the massive number of cancelations that occur upon multiplying B's with v's, which could be done from the very beginning in the formula but which would then obscure the meaning of the two terms. The reader is again warned that in the literature it is more common to use a different Jack normalization if one is interested in combinatoric formulae for which the other normalization is more suitable.

Appendix C. Coefficient B_1

The partition μ^{B_1} defined previously has the following properties: $\mu^{B_1}_i = \lambda_i, \forall i = 1, \ldots, N-k-1$ and $\mu^{B_1}_{N-k} = \lambda_{N-k} - 2 = r-2$. The conjugate partition has the

following properties: $\mu_i^{B_1'} = \lambda_i' \forall i = 1, \ldots, r-2$ and $i = r+1, \ldots, \frac{r}{k}(N-k)$ and $\mu_{r-1}^{B_1'} = \mu_r^{B_1'} = \lambda_{r-1}' - 1 = \lambda_r' - 1 = N-k-1$. We then have

$$\begin{aligned}
\frac{v_{\mu^{B_1}\mu^{B_1}}}{v_{\lambda\lambda}} &= \frac{\prod_{i=1}^{N-k-1}\prod_{j=1}^{\mu_i^{B_1}} h_*^{\mu^{B_1}}(i,j)}{\prod_{i=1}^{N-k-1}\prod_{j=1}^{\lambda_i} h_*^{\lambda}(i,j)} \frac{\prod_{j=1}^{\mu_{N-k}^{B_1}} h_*^{\mu^{B_1}}(N-k,j)}{\prod_{j=1}^{\lambda_{N-k}} h_*^{\lambda}(N-k,j)} \\
&= \prod_{i=1}^{N-k-1} \frac{h_*^{\mu^{B_1}}(i,r-1)h_*^{\mu^{B_1}}(i,r)}{h_*^{\lambda}(i,r-1)h_*^{\lambda}(i,r)} \cdot \prod_{j=1}^{r-2} \frac{h_*^{\mu^{B_1}}(N-k,j)}{h_*^{\lambda}(N-k,j)} \frac{1}{h_*^{\lambda}(N-k,r-1)h_*^{\lambda}(N-k,r)} \\
&= \prod_{i=1}^{N-k-1} \frac{h_*^{\mu^{B_1}}(i,r-1)h_*^{\mu^{B_1}}(i,r)}{h_*^{\lambda}(i,r-1)h_*^{\lambda}(i,r)} \cdot \prod_{j=1}^{r-2} \frac{1+\alpha(r-2-j)}{1+\alpha(r-j)} \frac{1}{1+\alpha} \\
&= \prod_{i=1}^{N-k-1} \frac{h_*^{\mu^{B_1}}(i,r-1)h_*^{\mu^{B_1}}(i,r)}{h_*^{\lambda}(i,r-1)h_*^{\lambda}(i,r)} \frac{1}{(1+\alpha(r-1))(1+\alpha(r-2))}. \qquad \text{(C.1)}
\end{aligned}$$

Now for B's:

$$\frac{\prod_{(i,j)\in\lambda} B^{\lambda}_{\lambda,\mu^{B_1}}(i,j)}{\prod_{(i,j)\in\mu^{B_1}} B^{\mu^{B_1}}_{\lambda,\mu^{B_1}}(i,j)} = \frac{\prod_{i=1}^{N-k-1}\prod_{j=1}^{\lambda_i} B^{\lambda}_{\lambda,\mu^{B_1}}(i,j)}{\prod_{i=1}^{N-k-1}\prod_{j=1}^{\mu_i^{B_1}} B^{\mu^{B_1}}_{\lambda,\mu^{B_1}}(i,j)} \frac{\prod_{j=1}^{\lambda_{N-k}} B^{\lambda}(N-k,j)}{\prod_{j=1}^{\mu_{N-k}^{B_1}} B^{\mu^{B_1}}(N-k,j)} \qquad \text{(C.2)}$$

$$\begin{aligned}
&= \prod_{i=1}^{N-k-1} \frac{B^{\lambda}(i,r-1)B^{\lambda}(i,r)}{B^{\mu^{B_1}}(i,r-1)B^{\mu^{B_1}}(i,r)} \prod_{j=1}^{r-2} \frac{B^{\lambda}(N-k,j)}{B^{\mu^{B_1}}(N-k,j)} B^{\lambda}(N-k,r-1)B^{\lambda}(N-k,r) \\
&= \prod_{i=1}^{N-k-1} \frac{h_*^{\lambda}(i,r-1)h_*^{\lambda}(i,r)}{h_*^{\mu^{B_1}}(i,r-1)h_*^{\mu^{B_1}}(i,r)} \prod_{j=1}^{r-2} \frac{h_{\lambda}^{*}(N-k,j)}{h_{\mu^{B_1}}^{*}(N-k,j)} h_*^{\lambda}(N-k,r-1)h_*^{\lambda}(N-k,r) \\
&= \prod_{i=1}^{N-k-1} \frac{h_*^{\lambda}(i,r-1)h_*^{\lambda}(i,r)}{h_*^{\mu^{B_1}}(i,r-1)h_*^{\mu^{B_1}}(i,r)} \prod_{j=1}^{r-2} \frac{r-j+1}{r-j-1}(1+\alpha) \\
&= \prod_{i=1}^{N-k-1} \frac{h_*^{\lambda}(i,r-1)h_*^{\lambda}(i,r)}{h_*^{\mu^{B_1}}(i,r-1)h_*^{\mu^{B_1}}(i,r)} \frac{r(r+1)}{2}(1+\alpha). \qquad \text{(C.3)}
\end{aligned}$$

By multiplying v's and B's and canceling the common factor (which again could have been canceled earlier), one gets

$$B_1 = \frac{r(r-1)}{2} \frac{1+\alpha}{(1+\alpha(r-1))(1+\alpha(r-2))}. \qquad \text{(C.4)}$$

Appendix D. Some Jack polynomial identities

Lassalle [23] found the following identity for Jack polynomials (the normalization in Lassale's paper is different than the normalization we use, so this formula has been modified appropriately):

$$\sum_i \frac{\partial}{\partial z_i} J^{\alpha}_{\{\lambda\}} = \sum_m A_{\{\lambda\},\{\lambda_{(m)}\}} J^{\alpha}_{\{\lambda_{(m)}\}}, \qquad \text{(D.1)}$$

where the coefficient reads

$$A_{\{\lambda\},\{\lambda_{(m)}\}} = \frac{1}{\alpha}\left(\prod_{j=m+1}^{l_\lambda}\frac{\alpha(\lambda_m-\lambda_j)+j-m-1}{\alpha(\lambda_m-\lambda_j)+j-m}\right)\left(\prod_{j=1}^{\lambda_m-1}\frac{\lambda'_j-m+1+\alpha(\lambda_m-j-1)}{\lambda'_j-m+1+\alpha(\lambda_m-j)}\right)$$
$$\times(l_\lambda-m+\alpha\lambda_m)(N-m+1+\alpha(\lambda_m-1)), \tag{D.2}$$

with $\lambda_{(m)}$ being the partition (elements of partitions are denoted by λ_i, but $\lambda_{(m)}$ is a full partition), where we remove 1 from the row λ_m in the partition λ, with l_λ being the length of the partition λ, and λ' being the partition conjugate to λ—this means that we write the partition $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_N)$ as a Young diagram with the ith row of length λ_i, and then transpose (as in matrix transposition) this to get λ'. As another example, $\lambda_{(m)}$ is the partition where 1 is subtracted from λ_m in λ : if $\lambda=(4,4,2,2)$ then $\lambda_{(1)}$ does not exist because the partition one would obtain is then $(3,4,2,2)$ which does not satisfy the rule that the partition must be made of decreasing integers. Then $\lambda_{(2)}=(4,3,2,2)$, $\lambda_{(3)}$ does not exist, and $\lambda_{(4)}=(4,4,2,1)$.

It is now easy to prove that

$$\sum_i\frac{\partial}{\partial z_i}J_{0k0^{r-1}k0^{r-1}k\cdots k0^{r-1}k}=J_{1k-10^{r-1}k0^{r-1}k\cdots k0^{r-1}k}, \tag{D.3}$$

First, let us translate everything in partition language

$$[0k0^{r-1}k0^{r-1}k\cdots k0^{r-1}k]\to\lambda\left(\underbrace{\frac{r}{k}(N-k)+1}_{k\text{ times}},\underbrace{\frac{r}{k}(N-k)+1-r}_{k\text{ times}},\ldots,\underbrace{2r+1}_{k\text{ times}},\underbrace{r+1}_{k\text{ times}},\underbrace{1}_{k\text{ times}}\right). \tag{D.4}$$

First, let us find the coefficient of $J_{1(k-1)0^{r-1}k0^{r-1}k\cdots k0^{r-1}k}$ in $\sum_i\frac{\partial}{\partial z_i}J_{0k0^{r-1}k0^{r-1}k\cdots k0^{r-1}k}$.

We have $l_\lambda=N$. In Lassale's notation, $1k-10^{r-1}k0^{r-1}k\cdots k0^{r-1}k$ corresponds to $\lambda_{(N)}$, i.e., the partition where 1 was subtracted from $\lambda_N=1$, the first two products in equation (D.2) do not contribute so:

$$A_{\{\lambda\},\{\lambda_{(N)}\}}=\frac{1}{\alpha}(N-N+\alpha\lambda_N)(N-N+1+\alpha(\lambda_N-1))=1. \tag{D.5}$$

Hence, we prove that the coefficient is 1, as we wanted. To prove equation (D.3), we also need to prove that all other contributions vanish. That is, we have a bunch of other $\lambda_{(m)}$'s which we can write as

$$\lambda_{(N-pk)}=\left(\underbrace{\frac{r}{k}(N-k)+1}_{k\text{ times}},\underbrace{\frac{r}{k}(N-k)+1-r}_{k\text{ times}},\ldots,\underbrace{(p+1)r}_{k\text{ times}},\underbrace{pr+1}_{k-1\text{ times}},\right.$$
$$\left.\times\,pr,\underbrace{(p-1)r}_{k\text{ times}},\ldots,\underbrace{2r+1}_{k\text{ times}},\underbrace{r+1}_{k\text{ times}},\underbrace{1}_{k\text{ times}}\right) \tag{D.6}$$

where p is an integer in the interval $\left[0,\ldots,\frac{N}{k}-1\right]$. In the original partition λ, the component $\lambda_{N-pk}=pr+1$. It is now easy to see that the coefficient $A_{\{\lambda\},\{\lambda_{(N-pk)}\}}=0$ for any p. The reason is that the second product on the rhs in the first row of equation (D.2) vanishes. The key is the numerator:

$$\prod_{j=1}^{\lambda_m-1}(\lambda'_j-m+1+\alpha(\lambda_m-j-1))=\prod_{j=1}^{\lambda_{N-pk}-1}(\lambda'_j-(N-pk)+1+\alpha(\lambda_{N-pk}-j-1))$$
$$=\prod_{j=1}^{pr}(\lambda'_j-(N-pk)+1+\alpha(pr-j)). \tag{D.7}$$

We now want to look at the term in the product that has $j_0 = (p-1)r + 1$. One can immediately see that $\lambda'_{j_0} = N - (p-1)k$. The product above then becomes

$$N - (p-1)k - (N - pk) + 1 + \alpha(pr - (p-1)r - 1) = k + 1 + \alpha(r-1) = 0, \tag{D.8}$$

where we have used the fact that we are looking at Jacks with $\alpha = -(k+1)/(r-1)$. As such we have proved equation (D.3). But we also know that $J_{0k0^{r-1}k0^{r-1}k\cdots k0^{r-1}k} = \prod_{i=1}^{N} z_i J_{k0^{r-1}k0^{r-1}k\cdots k0^{r-1}k}$ and hence we obtain

$$\sum_i \frac{\partial}{\partial z_i} J_{0k0^{r-1}k0^{r-1}k\cdots k0^{r-1}k} = (J_{1(k-1)0^{r-1}k0^{r-1}k\cdots k0^{r-1}k})$$

$$= \sum_i \frac{\partial}{\partial z_i} \prod_{j=1}^{N} z_j J_{k0^{r-1}k0^{r-1}k\cdots k0^{r-1}k} \tag{D.9}$$

$$= \sum_i \frac{1}{z_i} J_{0k0^{r-1}k0^{r-1}k\cdots k0^{r-1}k}, \tag{D.10}$$

where we have used the fact that $\sum_i \frac{\partial}{\partial z_i} J_{k0^{r-1}k0^{r-1}k\cdots k0^{r-1}k} = 0$ by virtue of $J_{k0^{r-1}k0^{r-1}k\cdots k0^{r-1}k}$ being a *highest weight* translationally invariant ground state [6].

Appendix E. Further Jack identities

Proposition 5.1 of [20] states that

$$J^{\alpha}_{\lambda}(z_1, \ldots, z_N) = \prod_i z_i J^{\alpha}_{\lambda - I}(z_1, \ldots, z_N), \tag{E.1}$$

where $\lambda - I = (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_n - 1)$ where n is the length of the partition. This of course supports that $\lambda_n > 0$ which means that the zeroth orbital, in occupation number language, must be zero. This proves our equations (60)–(62).

We then have (the sum over the particles i goes from $k+1$ to N, but the number of particles is explicit in the occupation number of any partition)

$$\sum_i \frac{\partial}{\partial z_i} J^{\alpha}_{0^r k0^{r-1}k0^{r-1}\cdots k0^{r-1}k} = \sum_i \frac{\partial}{\partial z_i} \prod_i z_i^r J^{\alpha}_{k0^{r-1}k0^{r-1}\cdots k0^{r-1}k}$$

$$= r \sum_i \frac{1}{z_i} \prod_i z_i^r J^{\alpha}_{k0^{r-1}k0^{r-1}\cdots k0^{r-1}k} + \prod_i z_i^r \sum_i \frac{\partial}{\partial z_i} J^{\alpha}_{k0^{r-1}k0^{r-1}\cdots k0^{r-1}k}$$

$$= r \prod_i z_i^{r-1} \sum_i \frac{1}{z_i} J^{\alpha}_{0k0^{r-1}k0^{r-1}\cdots k0^{r-1}k} \tag{E.2}$$

Now by equations (D.9), (D.10) and (D.3) in this paper, we get

$$\prod_i z_i^{r-1} \sum_i \frac{1}{z_i} J^{\alpha}_{0k0^{r-1}k0^{r-1}\cdots k0^{r-1}k} = \prod_i z_i^{r-1} J^{\alpha}_{1k-10^{r-1}k0^{r-1}\cdots k0^{r-1}k}$$

$$= J^{\alpha}_{0^{r-1}1k-10^{r-1}k0^{r-1}\cdots k0^{r-1}k} \tag{E.3}$$

so

$$\sum_i \frac{\partial}{\partial z_i} J^{\alpha}_{0^r k0^{r-1}k0^{r-1}\cdots k0^{r-1}k} = r J^{\alpha}_{0^{r-1}1k-10^{r-1}k0^{r-1}\cdots k0^{r-1}k}. \tag{E.4}$$

Note added. Since the completion of this manuscript, we have received a paper from Estienne, Regnault and Santachiara [47] who consider polynomials with (k, r) clustering which are not Jacks but can be described by unitary CFTs. This is possible because for $k = 2, r > 4$ and $k > 2, r > 2$, the thermodynamic limit Jacks are not the unique translationally invariant polynomials one can build with the (k, r) clustering property [6, 18]. The central charge of any (k, r) clustered polynomial can be computed by the method presented here.

References

[1] For a classic review of quantum Hall physics, see Prange R and Girvin S M (ed) 1987 *The Quantum Hall Effect* (New York: Springer)

[2] For a recent general review, see Nayak C, Simon S H, Stern A, Freedman M and DasSarma S 2008 Non-Abelian anyons and topological quantum computation *Rev. Mod. Phys.* **80** 1083 (http://www.arxiv.org/abs/0707.1889 and references therein)

[3] Di Francesco P , Mathieu P and Senechal D 1997 *Conformal Field Theory* (New York: Springer)

[4] Moore G and Read N 1991 Non-Abelions in the fractional quantum Hall effect *Nucl. Phys.* B **360** 362

[5] Read N and Rezayi E 1999 Beyond paired quantum Hall states: parafermions and incompressible states in the first excited Landau level *Phys. Rev.* B **59** 8084

[6] Bernevig B A and Haldane F D M 2008 Model fractional quantum Hall states and Jack polynomials *Phys. Rev. Lett.* **100** 246802

[7] Bernevig B A and Haldane F D M 2008 Generalized clustering conditions of Jack polynomials at negative Jack parameter *Phys. Rev.* B **77** 184502

[8] Bernevig B A and Haldane F D M 2008 Properties of non-Abelian fractional quantum Hall states at filling $\nu = k/r$ arXiv:0803.2882

[9] Jack H 1970 A class of symmetric polynomials with a parameter *Proc. R. Soc. Edinburgh* A **69** 1–18

[10] Green D 2002 *PhD Thesis* arXiv:cond-mat/0202455

[11] Wen X-G and Wu Y-S 1994 Chiral operator product algebra hidden in certain fractional quantum Hall wavefunctions *Nucl. Phys.* B **419** 45

[12] Simon S H, Rezayi E H, Cooper N R and Berdnikov I 2007 Construction of a paired wavefunction for spinless electrons at filling fraction $\nu = 2/5$ *Phys. Rev.* B **75** 075317

[13] Feigin B, Jimbo M, Miwa T and Mukhin E 2002 A differential ideal of symmetric polynomials spanned by Jack polynomials at $\beta = -(r-1)/(k+1)$ *Int. Math. Res. Not.* **2002** 1223 (arXiv:math/0112127)

[14] Feigin B, Jimbo M, Miwa T and Mukhin E 2003 Symmetric polynomials vanishing on the shifted diagonals and Macdonald polynomials *Int. Math. Res. Not.* **2003** 1015 (arXiv:math/0209042)

[15] Milovanovic M V, Jolicur Th and Vidanovic I 2009 Healing non-unitary conformal field theories and related fractional quantum Hall states arXiv:0902.1719

[16] Bernevig B A and Haldane F D M 2009 Clustering properties and model wavefunctions for non-Abelian fractional quantum hall quasielectrons *Phys. Rev. Lett.* **102** 066802 (arXiv:0810.2366)

[17] Regnault N, Bernevig B A and Haldane F D M 2009 Topological entanglement and clustering of Jain Hierarchy states arXiv:0901.2121

[18] Simon S H, Rezayi E H and Cooper N R 2007 Pseudopotentials for multiparticle interactions in the quantum Hall regime *Phys. Rev.* B **75** 195306

[19] Haldane F D M 2006 *Bull. Am. Phys. Soc.* **51** 633

[20] Stanley R P 1989 Some combinatorial properties of Jack symmetric functions *Adv. Math.* **77** 76

[21] Demmel J and Koev P 2006 Accurate and efficient evaluation of Schur and Jack functions *Math. Comput.* **75** 223

[22] Macdonald I G 1995 *Symmetric Functions and Hall Polynomials* 2nd edn (New York: Oxford University Press)

[23] Lassalle M 1998 Coefficients binomiaux generalises et polynomes de Macdonald *J. Funct. Anal.* **158** 289

[24] Read N 2008 Non-Abelian adiabatic statistics and Hall viscosity in quantum Hall states and $p_x + ip_y$ paired superfluids arXiv:0807.3107

[25] Read N 2007 Conformal invariance of chiral edge theories arXiv:0711.0543

[26] Zamolodchikov A and Fateev V 1985 Nonlocal (Parafermion) currents in two-dimensional conformal quantum field theory and self-dual critical points in Z_N-symmetrical statistical systems *Sov. Phys.—JETP* **62** 215–25

[27] Jacob P and Mathieu P 2005 The $Z_k^{(su(2),3/2)}$ parafermions *Phys. Lett.* B **627** 224

[28] Ravanini F 1992 On the possibility of Z_N exotic supersymmetry in two-dimensional conformal field theory *Int. J. Mod. Phys.* A **7** 4949

[29] Simon S H, Rezayi E H and Cooper N R 2007 Generalized quantum Hall projection Hamiltonians *Phys. Rev.* B **75** 075318

[30] Bergholtz E J, Hansson T H, Hermanns M and Karlhede A 2007 Microscopic theory of the quantum Hall Hierarchy *Phys. Rev. Lett.* **99** 256803
[31] Bergholtz E J, Hansson T H, Hermanns M, Karlhede A and Viefers S 2008 Hierarchy wavefunctions—from conformal correlators to Tao-Thouless states *Phys. Rev.* B **77** 165325
[32] Seidel A and Lee D-H 2006 Abelian and non-Abelian Hall liquids and charge-density wave: quantum number fractionalization in one and two dimensions *Phys. Rev. Lett.* **97** 056804
[33] Ardonne E, Bergholtz E J, Kailasvuori J and Wikberg E 2008 Degeneracy of non-Abelian quantum Hall states on the torus: domain walls and conformal field theory *J. Stat. Mech.* **2008** P04016
[34] Ardonne E 2008 Domain walls, fusion rules and conformal field theory in the quantum Hall regime arXiv:0809.0389
[35] Bernevig B A and Regnault N 2009 The anatomy of Abelian and non-Abelian fractional quantum Hall states arXiv:0902.4320
[36] Gurarie V and Nayak C 1997 A plasma analogy and berry matrices for non-Abelian quantum Hall states *Nucl. Phys.* B **506** 685
[37] Nayak C and Wilczek F 1996 $2n$ quasihole states realize 2^{n-1}-dimensional spinor braiding statistics in paired quantum Hall states *Nucl. Phys.* B **479** 529
[38] A good review of $\mathcal{W}$-symmetry in conformal field theory is given by Bouwknegt P and Schoutens K 1993 W symmetry in conformal field theory *Phys. Rep.* **223** 183
[39] Fateev V A and Lykyanov S L 1988 The models of two-dimensional conformal quantum field theory with Z_n symmetry *Int. J. Mod. Phys.* A **3** 507
[40] Fateev V A and Lykyanov S L 1990 Additional symmetries and exactly solvable models in two-dimensional conformal field theory *Sov. Sci. Rev. A. Phys.* **15** 1
[41] Bais F A, Bouwknegt P, Schoutens K and Surridge M 1988 Extensions of the Virasoro algebra constructed from Kac–Moody algebras using higher order Casimir invariants *Nucl. Phys.* B **304** 348
Bais F A, Bouwknegt P, Schoutens K and Surridge M 1988 Extensions of the Virasoro algebra constructed from Kac–Moody algebras using higher order Casimir invariants *Nucl. Phys.* B **304** 371
[42] Mizoguchi S 1989 Non-unitarity theorem for the A type W_n algebra *Phys. Lett.* B **231** 112
[43] Dunning C 2002 Massless flows between minimal W models *Phys. Lett.* B **537** 297
[44] Froehlich J, Pedrini B, Schweigert C and Walcher J 2004 Universality in quantum Hall systems: coset construction of incompressible states *J. Stat. Phys.* **103** 527
[45] See, for example, Forrester P J 1996 Jack polynomials and the multi-component Calogero–Sutherland model *Int. J. Mod. Phys.* B **10** 427 (and therein)
[46] Gurarie V, Flohr M and Nayak C 1997 The Haldane-Rezayi quantum Hall state and conformal field theory *Nucl. Phys.* B **498** 513
[47] Estienne B, Regnault N and Santachiara R 2009 arXiv:0904.3702